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Markov branching processes with interaction

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Abstract. In the paper we consider Markov processes, with countable set of states, interpreted as systems of particles of several types that interact as complexes for which the result of interaction with a complex of particles does not depend on the presence of other particles in the system. The apparatus of multivariate generating functions is used to find exact closed solutions of the first and second Kolmogorov system of differential equations for the transition probabilities. In our examples analytic methods are used to treat actual transmutation processes of particles in diverse areas of science.

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Introduction

(1) Differential equations remain the basic mathematical apparatus of modern science. The consideration of deterministic processes or changes in the *macroscopic* characteristics of physical phenomena in the course of time is a condition for applicability of this technique.

Probabilistic considerations arose in science in the second half of the nineteenth century and developed under the *microscopic* approach to physical processes, first of all in gases and chemical reactions. The molecular-kinetic theory considers, say, a gas as an aggregate of a vast number of chaotically moving particles (molecules, atoms, and so on) interacting with one another; the interaction takes place through collision of particles or by means of diverse forces. The basic conditions imposed

on the probabilistic approach were clarified, and the primary task can be formulated as follows: assuming that the laws regulating the behaviour of the particles forming the system are known, the problem is to establish the laws of behaviour of a macroscopic amount of matter by passing to the limit of a large number of particles, in particular, to establish the phenomenological laws connecting directly observable quantities (pressure, volume, concentration of reagents, and so on) [102]. A number of physical experiments carried out in the first third of the twentieth century proved the probabilistic nature of the microcosm, which finally led to the understanding that an adequate mathematical apparatus had to be constructed. "After the molecular viewpoint about the structure of matter became prevalent in physics, the appearance of new statistical (or probabilistic) methods of investigation in physical theories became inevitable" ([65], Russian p. 7).

The problem of constructing the foundations of the theory of stochastic processes was solved in the 1930s and 40s, mainly by Soviet mathematicians. The subsequent development of the theory of stochastic processes has been associated in large degree with the study of probability-theoretic schemes introduced in this period.

The application of probabilistic approaches to the study of processes in nature was characterized by a gradual passage from deterministic mechanistic concepts to probabilistic ideas. Originally, the equations of mechanics were supplemented by probabilistic hypotheses that were alien to mechanics itself (see the analysis of the Boltzmann kinetic equation in [79]). The subsequent success in the description of physical processes by the new methods involved the rejection of some deterministic concepts and the introduction of probabilistic definitions. Mathematical schemes of the theory of stochastic processes are completely free of the mechanistic assumptions, and this can explain both the universal applicability of these schemes in diverse areas of science and the depth of the analytic methods developed for them. These probability-theoretic schemes are based on the notion of a probability space $(\Omega, \mathcal{A}, \mathsf{P})$ of events.

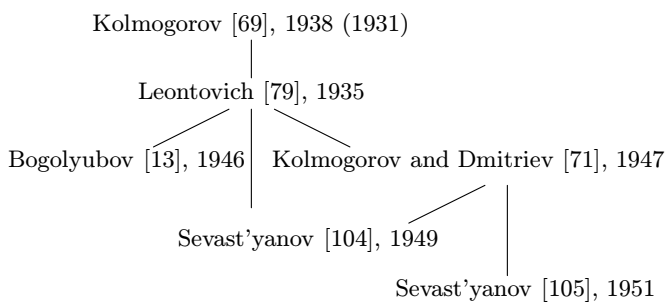
The progress achieved during the last decades in understanding the nature of the microcosm and the appearance of tools for investigating the molecular structure of matter led to the practical necessity of mathematically describing specific microscopic processes. Fairly many non-rigorously defined models (meeting the requirements for narrow classes of physical phenomena) were introduced in which elements of the deterministic and probabilistic approaches were combined. These models are frequently incompatible with the notion of a space of events; moreover, they are limited in the application of analytic methods. Models based on classical probability-theoretic schemes involving some notions of mechanical origin have also been investigated (for instance, when describing an interaction, the potential is sometimes replaced by a random potential [17], random forces are defined, and so on). In these models the passage to the limit in probabilistic schemes can lead to deterministic relationships. Some difficulties in applications indicate the incompleteness of the theory of stochastic processes; in particular, the necessity of introducing an interaction in probability-theoretic schemes has been repeatedly noted [110].

The development of the theory of stochastic processes has been determined by the fundamental concepts of the theory and by consideration of the above fundamental problems on this basis (though the physical knowledge that serves as a basis for

the probabilistic approach in science is insufficient at present). The objective of this paper is to show that possible steps in this direction can be based on the probability-theoretic schemes of the initial period of development of the theory of stochastic processes and on the existing analytic methods of the theory.

(2) In this paper we consider a generalization of Markov birth-and-death processes with countable state space $\mathbb{N}^n$ ($\mathbb{N} = \{0, 1, 2, \dots\}$) and with continuous time $t \in [0, \infty)$, which were introduced in [79], [104]. A point $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ of the state space is interpreted as a state of some physical system in which there is an aggregate of particles $S_\alpha = \alpha_1 T_1 + \dots + \alpha_n T_n$ that consists of α_1 particles of type T_1, ..., and α_n particles of type T_n; a passage of the stochastic process to another state is a result of interaction within some complex S_{ε^i}, $\varepsilon^i \in A$, of particles, where $A = \{\varepsilon^1, \dots, \varepsilon^l\} \subset \mathbb{N}^n$ is a given set. Such processes simulate a broad class of actual systems of interacting particles in physics, chemistry, and biology in which new particles appear as the result of interaction of several particles that exist at a given instant.

The general definition of Markov processes with interaction for a discrete state space has arisen by comparing several lines of research connected by a sequence of references [45]:



(0.1)

The interactions between the above papers are considered in Chapter 5 of the survey from the point of view of the following set of notions that arose in the study of probability-theoretic schemes introduced in (0.1): one-particle and many-particle distribution functions, independence conditions, the symmetry of distribution functions, the definition of interaction in terms of the independence conditions, a chain of equations, the state space of trajectories, and the kinetic equation for one-particle distribution functions. In the author's opinion (see [49], [50]), the analysis of the family of these notions in connection with Markov processes leads to the possibility of a certain step in the evolution of the theory of stochastic processes that is a consequence of the scheme (0.1). To support this, we construct explicit solutions of the Kolmogorov linear equations for Markov processes (see [46], [56]).

In this paper we present both known solutions of the Kolmogorov equations and new solutions. Markov branching processes with interaction are characterized by the application of generating functions to express the equations for the transition probabilities in the form of partial differential equations. The above results and statements of problems are aimed at deriving closed solutions of these partial differential equations. In the solution of the non-stationary and stationary Kolmogorov equations, expressions are obtained for the generating functions

of the desired probabilities in the form of series in special functions, and the main analytic complications have to do with summation of these series. The reduction of series to a closed form gives an integral representation for the generating functions in which the integrands are represented in a form having a probabilistic interpretation. We use orthogonal polynomials ([46], [47], [61], [62]), Bessel functions ([43], [46]), hypergeometric functions ([44], [59], [62]), and elliptic functions ([51], [61]).

Closed solutions of the Kolmogorov equations give simple proofs of certain limit theorems for Markov processes. Examples of limit theorems of this kind are presented in this paper.

The paper consists of five chapters. Every chapter is prefaced by a brief description of its contents. Chapter 1 contains the derivation of the main equations and forms a basis for the subsequent chapters. The exposition in each of the next three Chapters 2, 3, and 4 is independent of that in the other two chapters; the considerations in Chapter 5 are based on the results of Chapter 4. Chapters 2 and 5 are of a survey nature, whereas Chapters 3 and 4 are based on results of the author. We present proofs of theorems announced in our short communications [50], [48], [40] and give some new results.

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CHAPTER I

SPECIAL CLASSES OF MARKOV PROCESSES

The notion of special classes of Markov processes was introduced in [71], [104]. The class B_1 of branching processes was introduced in [71], and the class B_2 of branching processes with interaction was introduced in [104], where the relation

$$M \supset B_2 \supset B_1 \tag{1.1}$$

was noted (here M stands for the set of all Markov processes with states in $\mathbb{N}^n$). The relation (1.1) is called the *structure* of the set of Markov processes with the state space $\mathbb{N}^n$.

The material of § 1.1 and § 1.2 is of a preliminary nature. In §§ 1.3–1.6 we present the definitions of the classes B_1 and B_2 and some other special classes of homogeneous Markov processes with countable state space and continuous time. We give a complete and systematic presentation (see also § 2.1) of representations of the first and second Kolmogorov systems of partial differential equations for the transition probabilities with the help of multivariate generating functions and the operator of generalized differentiation (these results were partially published in [107], [108]). Corresponding to any class is a certain form of such partial differential equations. In § 1.7 we give a generalization of the structure (1.1). The results of Chapter 1 show that analytic methods for studying branching processes of class B_1 [106] can be extended to other classes of Markov processes.

§ 1.1. Markov processes on the state space $\mathbb{N}^n$

We denote by $\mathbb{N}^n = \{\alpha = (\alpha_1, \dots, \alpha_n),\ \alpha_i = 0, 1, 2, \dots,\ i = 1, \dots, n\}$ the set of all n-dimensional vectors with non-negative integral components. For $\alpha, \beta, \gamma \in \mathbb{N}^n$

we write $\gamma = \alpha - \beta$ if $\gamma_1 = \alpha_1 - \beta_1, \ldots, \gamma_n = \alpha_n - \beta_n$, $\alpha \geqslant \beta$ if $\alpha_1 \geqslant \beta_1, \ldots, \alpha_n \geqslant \beta_n$, and so on; we also set $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and denote the summation $\sum_{\alpha \in \mathbb{N}^n}$ by $\sum_\alpha$.

Let us list the main results of the theory of homogeneous Markov processes with countable state space ([30], Chap. 7, § 3, [18]). Let $\xi(t) = (\xi_1(t), \ldots, \xi_n(t))$ be a Markov process on the set $\mathbb{N}^n$ with continuous time $t \in [0, \infty)$. We denote the transition probabilities by

$$P_{\alpha\beta}(t) = \mathsf{P}\{\xi(t) = \beta \mid \xi(0) = \alpha\}, \qquad \alpha, \beta \in \mathbb{N}^n.$$

The condition that the process be time homogeneous means that

$$\mathsf{P}\{\xi(t_1 + t) = \beta \mid \xi(t_1) = \alpha\} = \mathsf{P}\{\xi(t) = \beta \mid \xi(0) = \alpha\}$$

for any $\alpha, \beta \in \mathbb{N}^n$ and $t_1, t \in [0, \infty)$. The transition probabilities of the process $\xi(t)$ satisfy the following conditions: $P_{\alpha\beta}(t) \geqslant 0$ for any $\alpha, \beta \in \mathbb{N}^n$, $t \in [0, \infty)$ (non-negativity condition); $\sum_\beta P_{\alpha\beta}(t) = 1$ for any $\alpha \in \mathbb{N}^n$, $t \in [0, \infty)$ (scaling condition);

$$P_{\alpha\beta}(t) = \sum_\gamma P_{\alpha\gamma}(t_1) P_{\gamma\beta}(t - t_1)$$

for any $\alpha, \beta \in \mathbb{N}^n$ and $0 \leqslant t_1 \leqslant t$, $t_1, t \in [0, \infty)$ (Markov property); $P_{\alpha\alpha}(0) = 1$ and $P_{\alpha\beta}(0) = 0$ $(\alpha \neq \beta)$ for any $\alpha, \beta \in \mathbb{N}^n$ (initial condition). It is assumed that the process $\xi(t)$ is stochastically continuous, that is, the conditions $\lim_{t \to 0+} P_{\alpha\alpha}(t) = 1$ and $\lim_{t \to 0+} P_{\alpha\beta}(t) = 0$ hold for any $\alpha \neq \beta$.

The following limits (which can be finite or infinite) always exist:

$$a_{\alpha\alpha} = \lim_{t \to 0+} \frac{P_{\alpha\alpha}(t) - 1}{t}, \quad a_{\alpha\beta} = \lim_{t \to 0+} \frac{P_{\alpha\beta}(t)}{t} \quad (\alpha \neq \beta), \qquad \alpha, \beta \in \mathbb{N}^n.$$

These limits are called the *infinitesimal characteristics* (or the *densities of the transition probabilities*) and are denoted by $a_{\alpha\beta} = (dP_{\alpha\beta}(t)/dt)|_{t=0+}$ for $\alpha, \beta \in \mathbb{N}^n$. If $\alpha \neq \beta$, then $a_{\alpha\beta}$ is finite; $a_{\alpha\alpha}$ is either finite or equal to $-\infty$; in any case $\sum_{\beta \neq \alpha} a_{\alpha\beta} \leqslant -a_{\alpha\alpha}$. A classification of the states of the process can be carried out by using the quantities $a_{\alpha\beta}$. A state α is said to be *instantaneous* if $a_{\alpha\alpha} = -\infty$. A non-instantaneous state is said to be *regular* if $\sum_{\beta \neq \alpha} a_{\alpha\beta} = -a_{\alpha\alpha}$.

The probabilistic meaning of the characteristics $\{a_{\alpha\beta}, \alpha, \beta \in \mathbb{N}^n\}$ is as follows. The Markov process $\xi(t)$ stays in an initial state α up to a random time τ_α with exponential distribution $\mathsf{P}\{\tau_\alpha \leqslant t\} = 1 - e^{a_{\alpha\alpha} t}$. If $a_{\alpha\alpha} = 0$, then the process cannot leave the state α (such a state is said to be *absorbing*). If $a_{\alpha\alpha} < 0$, then at the time τ_α the process goes to a state β with probability distribution $\{p_\beta^\alpha = -a_{\alpha\beta}/a_{\alpha\alpha}$, $\beta \neq \alpha$; $p_\alpha^\alpha = 0\}$; the process stays in the state β for a random time τ_β; and the further evolution of the process is similar.

1.1.1. First and second Kolmogorov systems of differential equations. If all states of a Markov process are regular, then the transition probabilities satisfy the system of equations

$$\frac{dP_{\alpha\beta}(t)}{dt} = \sum_\gamma a_{\alpha\gamma} P_{\gamma\beta}(t), \qquad \alpha \in \mathbb{N}^n, \tag{1.2}$$

with the initial conditions $P_{\alpha\alpha}(0) = 1$ and $P_{\alpha\beta}(0) = 0$ for $\alpha \neq \beta$. This is called the *first* (*backward*) Kolmogorov system of differential equations [69]. Let all states of the Markov process be regular and let $\sum_\gamma P_{\alpha\gamma}(t) a_{\gamma\gamma} > -\infty$ for a given α. Then for this α the *second* (*forward*) Kolmogorov system of differential equations [69] holds for the transition probabilities:

$$\frac{dP_{\alpha\beta}(t)}{dt} = \sum_\gamma P_{\alpha\gamma}(t)\, a_{\gamma\beta}, \qquad \beta \in \mathbb{N}^n, \tag{1.3}$$

with the initial conditions $P_{\alpha\alpha}(0) = 1$ and $P_{\alpha\beta}(0) = 0$ for $\alpha \neq \beta$.

Existence and uniqueness conditions for the solutions of the first and second system of equations are given in [26] (vol. 1, Chap. 17, § 9), [30], [18]. Under the above assumptions on the coefficients $a_{\alpha\beta}$ there always exists a unique *minimal solution*, that is, a non-negative solution satisfying both systems of differential equations (1.2) and (1.3), the Markov condition, and the relation $\sum_\beta P_{\alpha\beta}(t) \leqslant 1$. In particular, the minimal solution satisfies the identity $\sum_\beta P_{\alpha\beta}(t) \equiv 1$ for any $\alpha \in \mathbb{N}^n$ if the coefficients are bounded, that is, $|a_{\alpha\beta}| < C < \infty$ for any $\alpha, \beta \in \mathbb{N}^n$ [26]. Below we consider some Kolmogorov systems of equations for which a solution is not unique; in these cases we present an expression for the minimal solution.

§ 1.2. Multivariate generating functions

We denote by a_α the value at a point α of a numerical function a defined on $\mathbb{N}^n$. By the *multivariate generating function* $F(s_1, \dots, s_n)$ corresponding to $\{a_\alpha\}$ we mean the sum of the series

$$F(s_1, \dots, s_n) = \sum_\alpha a_\alpha s_1^{\alpha_1} \cdots s_n^{\alpha_n}.$$

By the *exponential multivariate generating function* $G(z_1, \dots, z_n)$ corresponding to $\{a_\alpha\}$ we mean the sum of the series

$$G(z_1, \dots, z_n) = \sum_\alpha \frac{z_1^{\alpha_1} \cdots z_n^{\alpha_n}}{\alpha_1! \cdots \alpha_n!} a_\alpha.$$

Below we also write briefly $s = (s_1, \dots, s_n)$, $s^\alpha = s_1^{\alpha_1} \cdots s_n^{\alpha_n}$, and $\alpha! = \alpha_1! \cdots \alpha_n!$. A vector of the form $s = (s_1, \dots, s_n)$ is denoted by 1 if all its components are equal to 1; we write $|s|$ for the vector with the components $|s_i|$ and use similar notation for a vector $z = (z_1, \dots, z_n)$. We set $zs = z_1 s_1 + \dots + z_n s_n$ and write

$$\frac{\partial^\alpha F(s)}{\partial s^\alpha} = \frac{\partial^{|\alpha|} F(s_1, \dots, s_n)}{\partial s_1^{\alpha_1} \cdots \partial s_n^{\alpha_n}}$$

and $\alpha^{[\beta]} = \alpha_1^{[\beta_1]} \cdots \alpha_n^{[\beta_n]}$, where $\alpha_i^{[\beta_i]} = \alpha_i(\alpha_i - 1) \cdots (\alpha_i - \beta_i + 1)$ for $i = 1, \dots, n$. If $F(s)$ is the generating function for $\{a_\alpha\}$, then $\partial^\beta F(s)/\partial s^\beta$ is the generating function for $\{\alpha^{[\beta]} a_\alpha,\ \alpha \in \mathbb{N}^n\}$. Let the generating functions $F(s)$ and $G(z)$ have non-empty domains of convergence. Then a one-to-one correspondence between the family $\{a_\alpha\}$ and each of the functions $F(s)$ and $G(z)$ is given by

$$a_\alpha = \frac{1}{\alpha!} \frac{\partial^\alpha F(0)}{\partial s^\alpha}, \qquad a_\alpha = \frac{\partial^\alpha G(0)}{\partial z^\alpha}.$$

A generating function is said to be *positive* if $a_\alpha \geqslant 0$ for $\alpha \in \mathbb{N}^n$. A positive generating function $F(s)$ is called a *probability generating function* if $F(1) = 1$. A probability generating function $F(s)$ corresponds to some probability distribution $\{a_\alpha\}$ on $\mathbb{N}^n$. A probability generating function can be assigned not only to a distribution $\{a_\alpha\}$ but also to a random vector $\xi = (\xi_1, \dots, \xi_n)$ whose probability distribution is $\{a_\alpha\}$. Using the vector ξ, we can give an equivalent distribution of a probability generating function, namely, $F_\xi(s) = \mathsf{E}s^\xi$. The expectation $\mathsf{E}\xi^{[\beta]}$ is called the *factorial moment* of order $|\beta| = \beta_1 + \dots + \beta_n$. It can be shown that

$$\mathsf{E}\xi^{[\beta]} = \frac{\partial^\beta F_\xi(1)}{\partial s^\beta}\,, \tag{1.4}$$

where the derivative at the point $s = 1$ is understood as the left derivative with respect to all the coordinates s_i, $i = 1, \dots, n$. In particular, $\mathsf{E}\xi_i = (\partial F_\xi(s)/\partial s_i)|_{s=1}$ for the expectations of the components of the random vector ξ, $i = 1, \dots, n$. The expression for the variance is

$$\mathsf{D}\xi_i = \frac{\partial^2 F_\xi(1)}{\partial s_i^2} + \frac{\partial F_\xi(1)}{\partial s_i} - \left(\frac{\partial F_\xi(1)}{\partial s_i}\right)^2, \qquad i = 1, \dots, n. \tag{1.5}$$

Multiplicative property. If $\xi^{(1)}, \dots, \xi^{(m)}$ are independent random vectors, then the generating function of their sum is equal to the product of the generating functions of the summands, that is,

$$F_{\xi^{(1)}+\dots+\xi^{(m)}}(s) = F_{\xi^{(1)}}(s) \cdots F_{\xi^{(m)}}(s). \tag{1.6}$$

In particular, if $\xi^{(1)}, \dots, \xi^{(m)}$ are independent identically distributed random vectors, then

$$F_{\xi^{(1)}+\dots+\xi^{(m)}}(s) = F^m_{\xi^{(1)}}(s). \tag{1.7}$$

For the proofs of the relations (1.4)–(1.7) and other results on multivariate generating functions, see [106], Chap. 4, § 1.

§ 1.3. Markov processes with interaction

Let $A = \{\varepsilon^i \in \mathbb{N}^n,\ i = 1, \dots, l\}$, be a finite set of vectors. To any vector ε^i we assign a probability distribution $\{p^i_\gamma \geqslant 0,\ \sum_\gamma p^i_\gamma = 1;\ p^i_{\varepsilon^i} = 0\}$ on $\mathbb{N}^n$ and a set $\{\varphi^i_\alpha \geqslant 0,\ \alpha \in \mathbb{N}^n;\ \varphi^i_\alpha = 0 \text{ if } \alpha_k < \varepsilon^i_k \text{ for some } k\}$ of numbers. For a Markov process with interaction $\xi(t)$ we define the *infinitesimal characteristics* by the relations

$$a_{\alpha\alpha} = -\sum_{i=1}^{l} \varphi^i_\alpha, \quad a_{\alpha\beta} = \sum_{i=1}^{l} \varphi^i_\alpha p^i_{\beta-\alpha+\varepsilon^i} \quad (\alpha \neq \beta), \quad \alpha, \beta \in \mathbb{N}^n. \tag{1.8}$$

Thus, a Markov process with interaction is determined by the set $\varepsilon^1, \{p^1_\gamma\}$, $\{\varphi^1_\alpha\}$, $\dots$, ε^l, $\{p^l_\gamma\}$, $\{\varphi^l_\alpha\}$.

We denote by M_1 the set of processes with interaction; thus, $M_1 \subset M$. We note that any Markov process with finitely many states belongs to the set M_1, and that an arbitrary Markov process in M can be interpreted as a process with infinite set

$A = \mathbb{N}^n$, where corresponding to every vector $\varepsilon \in \mathbb{N}^n$ is a probability distribution on $\mathbb{N}^n$ of the form $\{p^\varepsilon_\gamma = -a_{\varepsilon\gamma}/a_{\varepsilon\varepsilon},\ \gamma \neq \varepsilon;\ p^\varepsilon_\varepsilon = 0\}$ and the set $\{\varphi^\varepsilon_\alpha = 0,\ \alpha \neq \varepsilon;\ \varphi^\varepsilon_\varepsilon = -a_{\varepsilon\varepsilon}\}$ of numbers.

Special classes of Markov processes are singled out in the set M_1 by specifying a type of the functions $\varphi^1, \dots, \varphi^l$ (cf. § 2.1) defining the sets $\{\varphi^1_\alpha = \varphi^1(\alpha),\ \alpha \in \mathbb{N}^n\}$, $\dots$, $\{\varphi^l_\alpha = \varphi^l(\alpha),\ \alpha \in \mathbb{N}^n\}$ of numbers.

1.3.1. Interpretation in the form of a system with transmutations of particles. A physical interpretation of a Markov process $\xi(t)$ belonging to the set M_1 is as follows ([104], [16]). An event of the form $\{\xi(t) = \alpha\}$ can be treated as a state of a system in which at a time t there is an aggregate S_α of particles consisting of α_1 particles of type T_1, α_2 particles of type T_2, $\dots$, α_n particles of type T_n, that is, $S_\alpha = \alpha_1 T_1 + \alpha_2 T_2 + \dots + \alpha_n T_n$. Let us introduce l complexes S_{ε^i} of particle interaction that correspond to vectors $\varepsilon^i \in A$. An interaction of the complex S_{ε^i} of particles occurs in a random time τ^i_α such that $\mathsf{P}\{\tau^i_\alpha \leqslant t\} = 1 - e^{-\varphi^i_\alpha t}$. At this instant, ε^i_1 particles are chosen among the α_1 particles of type T_1, ε^i_2 particles are chosen among the α_2 particles of type T_2, $\dots$, ε^i_n particles are chosen among the α_n particles of type T_n, and this complex S_{ε^i} of particles is replaced by an aggregate S_γ of new particles with probability distribution $\{p^i_\gamma\}$. The system passes from the state S_α corresponding to the vector α to the state $S_{\alpha-\varepsilon^i+\gamma}$ corresponding to the vector $\alpha - \varepsilon^i + \gamma$, and the further evolution of the particle system is similar. The system stays in the state S_α during a random time τ_α until one of the l possible interactions occurs, that is, $\tau_\alpha = \min(\tau^1_\alpha, \dots, \tau^l_\alpha)$. It is assumed that the random variables $\tau^1_\alpha, \dots, \tau^l_\alpha$ are independent. In this case $\mathsf{P}\{\tau_\alpha \leqslant t\} = 1 - e^{-(\varphi^1_\alpha + \dots + \varphi^l_\alpha)t}$, and the probability of an interaction of the complex S_{ε^i} of particles (under the condition that some interaction indeed occurred) is equal to $\varphi^i_\alpha \big(\sum_{i=1}^l \varphi^i_\alpha\big)^{-1}$ ([12], Chap. 1, § 2). Let us represent the possible transmutations of particles in this system by the following *interaction scheme*:

$$\begin{cases} \varepsilon^1_1 T_1 + \varepsilon^1_2 T_2 + \dots + \varepsilon^1_n T_n \to \gamma^1_1 T_1 + \gamma^1_2 T_2 + \dots + \gamma^1_n T_n, \\ \qquad \dots \\ \varepsilon^i_1 T_1 + \varepsilon^i_2 T_2 + \dots + \varepsilon^i_n T_n \to \gamma^i_1 T_1 + \gamma^i_2 T_2 + \dots + \gamma^i_n T_n, \\ \qquad \dots \\ \varepsilon^l_1 T_1 + \varepsilon^l_2 T_2 + \dots + \varepsilon^l_n T_n \to \gamma^l_1 T_1 + \gamma^l_2 T_2 + \dots + \gamma^l_n T_n, \end{cases} \tag{1.9}$$

where the distribution of the random vector $\gamma^i = (\gamma^i_1, \gamma^i_2, \dots, \gamma^i_n)$ is $\{p^i_\gamma\}$ for any $i = 1, \dots, l$.

1.3.2. Second equation for the generating function of the transition probabilities. Let us consider the transition probabilities $P_{\alpha\beta}(t)$, $\alpha, \beta \in \mathbb{N}^n$, of a Markov process with interaction. We introduce the following multivariate generating functions:

$$F_\alpha(t; s) = \sum_\beta P_{\alpha\beta}(t) s^\beta, \quad \alpha \in \mathbb{N}^n, \qquad h_i(s) = \sum_\gamma p^i_\gamma s^\gamma, \quad i = 1, \dots, l, \ \ |s| < 1, \tag{1.10}$$

and note that $F_\alpha(t;s)$ is an analytic function of the variables $s_1,\dots,s_n$ in the domain under consideration because $|F_\alpha(t;s)| \leqslant \sum_\beta P_{\alpha\beta}(t)|s_1|^{\beta_1}\cdots|s_n|^{\beta_n} \leqslant \sum_\beta P_{\alpha\beta}(t) \leqslant 1$. Below we need the Gel'fond–Leont'ev operator of generalized differentiation ([29], [103]) defined on functions analytic in a neighbourhood of the origin:

$$D_i\biggl(\sum_\beta a_\beta s^\beta\biggr) = \sum_{\beta\geqslant\varepsilon^i} a_\beta\varphi^i_\beta s^{\beta-\varepsilon^i}, \qquad i=1,\dots,l.$$

By definition, all states of a Markov process with interaction are regular. Suppose that all the conditions of § 1.1.1 for the second system of equations are satisfied.

Proposition 1.1. *The second Kolmogorov system of differential equations for a Markov process with interaction can be represented in the concise form*

$$\frac{\partial F_\alpha(t;s)}{\partial t} = \sum_{i=1}^l \bigl(h_i(s)-s^{\varepsilon_i}\bigr)D_i(F_\alpha(t;s)), \qquad F_\alpha(0;s)=s^\alpha. \tag{1.11}$$

Proof. Substituting the expressions for the infinitesimal characteristics (1.8) into the system (1.3), we obtain the following chain of equalities:

$$\begin{aligned}
\frac{\partial F_\alpha(t;s)}{\partial t} &= \sum_\beta \frac{dP_{\alpha\beta}(t)}{dt}s^\beta = \sum_\beta\biggl(\sum_\gamma P_{\alpha\gamma}(t)a_{\gamma\beta}\biggr)s^\beta \\
&= \sum_{i=1}^l\biggl(\sum_{\gamma\geqslant\varepsilon^i}\sum_{\beta-\gamma+\varepsilon^i\geqslant 0} P_{\alpha\gamma}(t)\varphi^i_\gamma p^i_{\beta-\gamma+\varepsilon^i}s^\beta - \sum_{\beta\geqslant\varepsilon^i} P_{\alpha\beta}(t)\varphi^i_\beta s^\beta\biggr) \\
&= \sum_{i=1}^l\sum_{\gamma\geqslant\varepsilon^i} P_{\alpha\gamma}(t)\varphi^i_\gamma s^{\gamma-\varepsilon^i}\biggl(\sum_{\beta-\gamma+\varepsilon^i\geqslant 0} p^i_{\beta-\gamma+\varepsilon^i}s^{\beta-\gamma+\varepsilon^i} - s^{\varepsilon_i}\biggr) \\
&= \sum_{i=1}^l D_i(F_\alpha(t;s))\bigl(h_i(s)-s^{\varepsilon^i}\bigr).
\end{aligned}$$

This proves Proposition 1.1.

Below we treat the problem of the equivalence of the system of equations (1.3) and the equation (1.11) for specific sets $\{\varphi^1_\alpha\},\dots,\{\varphi^l_\alpha\}$ of numbers.

§ 1.4. Branching processes with interaction

A special class B_2 of Markov processes belonging to the set M_1 is singled out by the following conditions on $\{\varphi^1_\alpha\},\dots,\{\varphi^l_\alpha\}$. Let a set $A=\{\varepsilon^i=(\varepsilon^i_1,\dots,\varepsilon^i_n),\ i=1,\dots,l\}$, probability distributions $\{p^i_\gamma\}$, $i=1,\dots,l$, and sets

$$\varphi^i_\alpha = \lambda_i\prod_{j=1}^n \alpha_j(\alpha_j-1)\cdots(\alpha_j-\varepsilon^i_j+1) = \lambda_i\alpha^{[\varepsilon^i]}, \qquad \alpha\in\mathbb{N}^n,$$

be given, where the $\lambda_i>0$ are proportionality coefficients, $i=1,\dots,l$. The infinitesimal characteristics $\{a_{\alpha\beta},\ \alpha,\beta\in\mathbb{N}^n\}$ of a branching process with interaction are defined by the formulae (1.8).

The above choice of numbers $\{\varphi^i_\alpha\}$ was proposed in [104]. Let a Markov process be in some state $\alpha = (\alpha_1, \dots, \alpha_n)$, which corresponds to the presence of an aggregate S_α of particles. It is assumed that, in a time Δt, $\Delta t \to 0$, the probability $\varphi^i_\alpha \Delta t + o(\Delta t)$ of interaction of the complex S_{ε^i} of particles is proportional to the number $\binom{\alpha_1}{\varepsilon^i_1}$ of combinations selecting ε^i_1 particles of type T_1 out of the existing α_1 particles of type T_1, ..., and proportional to the number $\binom{\alpha_n}{\varepsilon^i_n}$ of combinations selecting ε^i_n particles of type T_n out of the existing α_n particles of type T_n.

1.4.1. First equation for the exponential generating function of transition probabilities. Let us introduce the exponential generating function of transition probabilities,

$$G_\beta(t; z) = \sum_\alpha \frac{z^\alpha}{\alpha!} P_{\alpha\beta}(t), \qquad \beta \in \mathbb{N}^n, \tag{1.12}$$

and the linear differential operators with constant coefficients

$$h_i\left(\frac{\partial}{\partial z}\right) = \sum_\gamma p^i_\gamma \frac{\partial^\gamma}{\partial z^\gamma}, \qquad i = 1, \dots, l.$$

The function $G_\beta(t; z)$ is analytic with respect to the variables $z_1, \dots, z_n$ for any β, because

$$|G_\beta(t; z)| \leqslant \sum_\alpha \frac{|z_1|^{\alpha_1} \cdots |z_n|^{\alpha_n}}{\alpha_1! \cdots \alpha_n!} P_{\alpha\beta}(t) \leqslant e^{|z_1| + \dots + |z_n|}. \tag{1.13}$$

Theorem 1.2 [108]. *The exponential generating function $G_\beta(t; z)$ of the transition probabilities for a branching process with interaction satisfies the following linear partial differential equation for any $\beta \in \mathbb{N}^n$:*

$$\frac{\partial G_\beta(t; z)}{\partial t} = \sum_{i=1}^l \lambda_i z^{\varepsilon^i} \left(h_i\left(\frac{\partial}{\partial z}\right) - \frac{\partial^{\varepsilon^i}}{\partial z^{\varepsilon^i}} \right) G_\beta(t; z), \qquad G_\beta(0; z) = \frac{z^\beta}{\beta!}. \tag{1.14}$$

Proof. To derive the equation (1.14), we reduce the first system of differential equations (1.2) for a branching process with interaction with the help of the multivariate generating function (1.12),

$$\frac{dP_{\alpha\beta}(t)}{dt} = \sum_{i=1}^l \sum_{\gamma - \alpha + \varepsilon^i \geqslant 0} \lambda_i \alpha^{[\varepsilon^i]} p^i_{\gamma - \alpha + \varepsilon^i} P_{\gamma\beta}(t) - \sum_{i=1}^l \lambda_i \alpha^{[\varepsilon^i]} P_{\alpha\beta}(t), \quad \alpha \in \mathbb{N}^n; \tag{1.15}$$

the initial conditions are $P_{\alpha\alpha}(0) = 1$ and $P_{\alpha\beta}(0) = 0$ for $\alpha \neq \beta$. It follows from (1.13) that the series (1.12) is absolutely and uniformly convergent with respect to t for any fixed z. The derivative $\partial G_\beta(t; z)/\partial t$ is equal to the sum of the series

$$\sum_\alpha \frac{z^\alpha}{\alpha!} \frac{dP_{\alpha\beta}(t)}{dt},$$

because the latter is absolutely and uniformly convergent with respect to t. Indeed, it follows from (1.15) that

$$\begin{aligned}
&\Bigl|\sum_{|\alpha|\leqslant L}\frac{z^\alpha}{\alpha!}\frac{dP_{\alpha\beta}(t)}{dt}\Bigr| \\
&\quad=\Bigl|\sum_{|\alpha|\leqslant L}\frac{z^\alpha}{\alpha!}\sum_{i=1}^{l}\sum_{\gamma-\alpha+\varepsilon^i\geqslant 0}\lambda_i\alpha^{[\varepsilon^i]}p^i_{\gamma-\alpha+\varepsilon^i}P_{\gamma\beta}(t)-\sum_{|\alpha|\leqslant L}\frac{z^\alpha}{\alpha!}\sum_{i=1}^{l}\lambda_i\alpha^{[\varepsilon^i]}P_{\alpha\beta}(t)\Bigr| \\
&\quad\leqslant\sum_{i=1}^{l}\lambda_i\sum_{\alpha\geqslant\varepsilon^i}\frac{|z|^\alpha}{(\alpha-\varepsilon^i)!}\sum_r p^i_r+\sum_{i=1}^{l}\lambda_i\sum_{\alpha\geqslant\varepsilon^i}\frac{|z|^\alpha}{(\alpha-\varepsilon^i)!} \\
&\quad=2e^{|z_1|+\dots+|z_n|}\sum_{i=1}^{l}\lambda_i|z|^{\varepsilon^i}<\infty,
\end{aligned}$$

and we can apply the Weierstrass M-test for uniform convergence of a series. These remarks justify the following formula:

$$\begin{aligned}
\frac{\partial G_\beta(t;z)}{\partial t}&=\sum_\alpha\frac{z^\alpha}{\alpha!}\frac{dP_{\alpha\beta}(t)}{dt}=\sum_\alpha\frac{z^\alpha}{\alpha!}\Bigl(\sum_\gamma a_{\alpha\gamma}P_{\gamma\beta}(t)\Bigr) \\
&=\sum_{i=1}^{l}\lambda_i z^{\varepsilon^i}\Bigl(\sum_{\alpha\geqslant\varepsilon^i}\sum_{\gamma-\alpha+\varepsilon^i\geqslant 0}\frac{z^{\alpha-\varepsilon^i}}{(\alpha-\varepsilon^i)!}p^i_{\gamma-\alpha+\varepsilon^i}P_{\gamma\beta}(t)-\sum_{\alpha\geqslant\varepsilon^i}\frac{z^{\alpha-\varepsilon^i}}{(\alpha-\varepsilon^i)!}P_{\alpha\beta}(t)\Bigr) \\
&=\sum_{i=1}^{l}\lambda_i z^{\varepsilon^i}\Bigl(\sum_r p^i_r\sum_{\gamma-r\geqslant 0}\frac{z^{\gamma-r}}{(\gamma-r)!}P_{\gamma\beta}(t)-\sum_{\alpha-\varepsilon^i\geqslant 0}\frac{z^{\alpha-\varepsilon^i}}{(\alpha-\varepsilon^i)!}P_{\alpha\beta}(t)\Bigr) \\
&=\sum_{i=1}^{l}\lambda_i z^{\varepsilon^i}\Bigl(\sum_r p^i_r\frac{\partial^r G_\beta(t;z)}{\partial z^r}-\frac{\partial^{\varepsilon^i}G_\beta(t;z)}{\partial z^{\varepsilon^i}}\Bigr) \\
&=\sum_{i=1}^{l}\lambda_i z^{\varepsilon^i}\Bigl(h_i\Bigl(\frac{\partial}{\partial z}\Bigr)-\frac{\partial^{\varepsilon^i}}{\partial z^{\varepsilon^i}}\Bigr)G_\beta(t;z).
\end{aligned}$$

The condition $G_\beta(0;z)=z^\beta/\beta!$ follows from the initial conditions for the system (1.15). This proves Theorem 1.2.

1.4.2. Second equation for the generating function of the transition probabilities. We use the multivariate generating functions (1.10).

Theorem 1.3 [104]. *The generating function $F_\alpha(t;s)$ of the transition probabilities of a branching process with interaction satisfies for any $\alpha\in\mathbb{N}^n$ the following linear partial differential equation for $|s|\leqslant 1$:*

$$\frac{\partial F_\alpha(t;s)}{\partial t}=\sum_{i=1}^{l}\lambda_i\bigl(h_i(s)-s^{\varepsilon^i}\bigr)\frac{\partial^{\varepsilon^i}F_\alpha(t;s)}{\partial s^{\varepsilon^i}},\qquad F_\alpha(0;s)=s^\alpha. \tag{1.16}$$

Proof. The second system of differential equations (1.3) for a branching process with interaction becomes

$$\frac{dP_{\alpha\beta}(t)}{dt} = \sum_{i=1}^{l} \sum_{\beta-\gamma+\varepsilon^i \geqslant 0} P_{\alpha\gamma}(t)\lambda_i \gamma^{[\varepsilon^i]} p^i_{\beta-\gamma+\varepsilon^i} - \sum_{i=1}^{l} P_{\alpha\beta}(t)\lambda_i \beta^{[\varepsilon^i]}, \quad \beta \in \mathbb{N}^n. \quad (1.17)$$

The series $F_\alpha(t;s) = \sum_\beta P_{\alpha\beta}(t)s^\beta$ is absolutely and uniformly convergent with respect to t for any fixed s, $|s| < 1$. The series $\sum_\beta (dP_{\alpha\beta}(t)/dt)s^\beta$ is absolutely and uniformly convergent with respect to t for $|s| < 1$, and hence the sum of this series is equal to $\partial F_\alpha(t;s)/\partial t$. Indeed, it follows from (1.17) that for $|s| < 1$ we have

$$\begin{aligned}
&\Bigg| \sum_{|\beta| \leqslant L} \frac{dP_{\alpha\beta}(t)}{dt} s^\beta \Bigg| \\
&\quad = \Bigg| \sum_{|\beta| \leqslant L} \sum_{i=1}^{l} \sum_{\beta-\gamma+\varepsilon^i \geqslant 0} P_{\alpha\gamma}(t)\lambda_i \gamma^{[\varepsilon^i]} p^i_{\beta-\gamma+\varepsilon^i} s^\beta - \sum_{|\beta| \leqslant L} \sum_{i=1}^{l} P_{\alpha\beta}(t)\lambda_i \beta^{[\varepsilon^i]} s^\beta \Bigg| \\
&\quad \leqslant \sum_{i=1}^{l} \lambda_i |s|^{\varepsilon^i} \sum_{\gamma \geqslant \varepsilon^i} \gamma^{[\varepsilon^i]} |s|^{\gamma-\varepsilon^i} \sum_r p^i_r + \sum_{i=1}^{l} \lambda_i |s|^{\varepsilon^i} \sum_{\beta \geqslant \varepsilon^i} \beta^{[\varepsilon^i]} |s|^{\beta-\varepsilon^i} \\
&\quad = 2\sum_{i=1}^{l} \lambda_i |s|^{\varepsilon^i} \prod_{j=1}^{n} \frac{1}{1-|s_j|} < \infty;
\end{aligned}$$

we can now apply the Weierstrass M-test for uniform convergence of a series. We obtain the equation (1.16) by using the system (1.17):

$$\begin{aligned}
\frac{\partial F_\alpha(t;s)}{\partial t} &= \sum_\beta \frac{dP_{\alpha\beta}(t)}{dt} s^\beta = \sum_\beta \Bigg(\sum_\gamma P_{\alpha\gamma}(t) a_{\gamma\beta} \Bigg) s^\beta \\
&= \sum_{i=1}^{l} \lambda_i \Bigg(\sum_{\gamma \geqslant \varepsilon^i} \sum_{\beta-\gamma+\varepsilon^i \geqslant 0} P_{\alpha\gamma}(t)\gamma^{[\varepsilon^i]} p^i_{\beta-\gamma+\varepsilon^i} s^\beta - \sum_{\beta \geqslant \varepsilon^i} P_{\alpha\beta}(t)\beta^{[\varepsilon^i]} s^\beta \Bigg) \\
&= \sum_{i=1}^{l} \lambda_i \sum_{\gamma \geqslant \varepsilon^i} P_{\alpha\gamma}(t)\gamma^{[\varepsilon^i]} s^{\gamma-\varepsilon^i} \Bigg(\sum_{\beta-\gamma+\varepsilon^i \geqslant 0} p^i_{\beta-\gamma+\varepsilon^i} s^{\beta-\gamma+\varepsilon^i} - s^{\varepsilon_i} \Bigg) \\
&= \sum_{i=1}^{l} \lambda_i \frac{\partial^{\varepsilon^i} F_\alpha(t;s)}{\partial s^{\varepsilon^i}} \big(h_i(s) - s^{\varepsilon^i} \big).
\end{aligned}$$

This proves the validity of (1.16) for $|s| < 1$. The validity of the equation for $|s| \leqslant 1$ follows from the continuity. This completes the proof of Theorem 1.3.

We introduce the double generating function

$$\mathcal{F}(t;z;s) = \sum_\alpha \frac{z^\alpha}{\alpha!} F_\alpha(t;s) = \sum_{\alpha,\beta} \frac{z^\alpha}{\alpha!} P_{\alpha\beta}(t) s^\beta = \sum_\beta G_\beta(t;z) s^\beta.$$

Corollary 1.4. *The generating function $\mathcal{F}(t;z;s)$ of the transition probabilities of a branching process with interaction satisfies the following equations for $|s|\leqslant 1$:*

$$\frac{\partial\mathcal{F}}{\partial t}=\sum_{i=1}^{l}\lambda_i z^{\varepsilon^i}\left(h_i\left(\frac{\partial}{\partial z}\right)-\frac{\partial^{\varepsilon^i}}{\partial z^{\varepsilon^i}}\right)\mathcal{F}, \tag{1.18}$$

$$\frac{\partial\mathcal{F}}{\partial t}=\sum_{i=1}^{l}\lambda_i\bigl(h_i(s)-s^{\varepsilon^i}\bigr)\frac{\partial^{\varepsilon^i}\mathcal{F}}{\partial s^{\varepsilon^i}}\,,\quad \mathcal{F}(0;z;s)=e^{zs}. \tag{1.19}$$

§ 1.5. Branching processes

A special class B_1 of Markov processes is singled out in the class B_2 by the conditions $|\varepsilon^i|\leqslant 1$, $i=1,\dots,l$. To be definite, let $A=\{\varepsilon^1=(1,0,\dots,0)$, $\varepsilon^2=(0,1,0,\dots,0)$, $\dots$, $\varepsilon^l=(0,\dots,0,1,0,\dots,0)\}$. Then $\varphi^i_\alpha=\lambda_i\alpha_i$, $\alpha\in\mathbb{N}^n$, $i=1,\dots,l$, and the second Kolmogorov equation (1.16) becomes

$$\frac{\partial F_\alpha(t;s)}{\partial t}=\sum_{i=1}^{l}\lambda_i\bigl(h_i(s)-s_i\bigr)\frac{\partial F_\alpha(t;s)}{\partial s_i}\,,\qquad F_\alpha(0;s)=s^\alpha. \tag{1.20}$$

The solution of the first-order partial differential equation (1.20) has the following *branching property* ([106], Chap. 4, § 2, (3)):

$$F_\alpha(t;s)=F^{\alpha_1}_{\varepsilon^1}(t;s)F^{\alpha_2}_{\varepsilon^2}(t;s)\cdots F^{\alpha_l}_{\varepsilon^l}(t;s)s_{l+1}^{\alpha_{l+1}}\cdots s_n^{\alpha_n},\qquad \alpha\in\mathbb{N}^n. \tag{1.21}$$

The branching property is the main feature distinguishing the class of branching processes among the Markov processes. It says that if a state of a process is interpreted as the existence of an aggregate of particles, then the particles existing at a moment t_1 evolve at any next moment t_1+t, $t>0$, and give new particles that are independent of one another. This follows from a comparison of the formula (1.21) with the properties (1.6) and (1.7) of the generating functions ([106], Chap. 4, § 2).

Theorem 1.5. *The generating functions $F_{\varepsilon^1}(t;s)$, $\dots$, $F_{\varepsilon^l}(t;s)$ of the transition probabilities of a branching process satisfy the following system of non-linear ordinary differential equations for $|s|\leqslant 1$:*

$$\begin{cases}\dfrac{\partial F_{\varepsilon^1}(t;s)}{\partial t}=\lambda_1\bigl(h_1(F_{\varepsilon^1}(t;s),\dots,F_{\varepsilon^l}(t;s),s_{l+1},\dots,s_n)-F_{\varepsilon^1}(t;s)\bigr),\\ \qquad\dots\\ \dfrac{\partial F_{\varepsilon^l}(t;s)}{\partial t}=\lambda_l\bigl(h_l(F_{\varepsilon^1}(t;s),\dots,F_{\varepsilon^l}(t;s),s_{l+1},\dots,s_n)-F_{\varepsilon^l}(t;s)\bigr),\end{cases} \tag{1.22}$$

with the initial conditions $F_{\varepsilon^1}(0;s)=s_1$, $\dots$, $F_{\varepsilon^l}(0;s)=s_l$.

The proof is based on the branching property (1.21) (see [106], Chap. 4, § 3, Theorem 3). For $n=1$ the non-linear equation is derived in § 5.1.

The particles of the types $T_{l+1},\dots,T_n$ are said to be *final* ([106], Chap. 5). If the vector $\varepsilon^0=(0,\dots,0)$ belongs to the set A, in which case $\varphi^0_\alpha=\lambda_0$ for any $\alpha\in\mathbb{N}^n$, then the branching process is called a process *with immigration* of particles ([106], Chap. 7).

A systematic exposition of the theory of branching stochastic processes is given in the monograph [106]. For a survey of results in the theory of branching processes, see [122].

§ 1.6. The class B_3

We define another class of Markov processes [42]. Suppose that a set $A = \{\varepsilon^1 = (1, 0, \dots, 0),\ \varepsilon^2 = (0, 1, 0, \dots, 0),\ \dots,\ \varepsilon^l = (0, \dots, 0, 1, 0, \dots, 0)\}$ is given, and corresponding to any vector ε^i is a probability distribution $\{p^i_\gamma\}$. Let us introduce sets $\{\varphi^i_\alpha\}$ of numbers as follows. Suppose that $U_i(x)$ is the distribution function of some non-negative random variable and that $U_i(x)$ is an infinitely divisible distribution function for any $i = 1, \dots, l$. It is known ([26], vol. 2, Chap. 13, § 7) that the Laplace transform of this function can be represented as

$$\int_0^\infty e^{-px}\, U_i\{dx\} = e^{-\psi_i(p)}, \qquad p \geqslant 0,$$

where $\psi_i(p) = \int_0^\infty x^{-1}(1 - e^{-px})\, P_i\{dx\}$ and P_i is a measure with $\int_1^\infty x^{-1} P_i\{dx\} < \infty$. We set $\varphi^i_\alpha = \psi_i(\alpha_i)$ for $\alpha \in \mathbb{N}^n$ and $i = 1, \dots, l$ and define the infinitesimal characteristics $\{a_{\alpha\beta},\ \alpha, \beta \in \mathbb{N}^n\}$ of a Markov process of class B_3 by the formulae (1.8). The choice of the numbers $\{\varphi^1_\alpha\}, \dots, \{\varphi^l_\alpha\}$ is related to properties (presented below in §§ 2.1.2 and § 5.4, see also [42]) of birth-and-death processes of power-law type. The class B_3 of processes is closest to the class B_1 of branching processes; indeed, if a distribution $U_i(x)$ is concentrated at a point λ_i ($\lambda_i > 0$), then $\varphi^i_\alpha = \lambda_i \alpha_i$ for $\alpha \in \mathbb{N}^n$ and $i = 1, \dots, l$, and we obtain a process of class B_1.

§ 1.7. Structure of the set of Markov processes

A generalization of the structure (1.1) for Markov processes with countable state spaces is of the form

$$\begin{array}{ccccc} M & \supset & M_1 & \supset & B_2 \\ & & \cup & & \cup \\ & & B_3 & \supset & B_1 \end{array} \qquad B_2 \cap B_3 = B_1, \tag{1.23}$$

where B_1 stands for the class of Markov branching processes and B_2 for the class of branching processes with interaction, the class B_3 is defined in § 1.6, and the set M_1 is described in § 1.3. The sets M_1, B_2, B_1, and B_3 were defined by indicating the infinitesimal characteristics $\{a_{\alpha\beta},\ \alpha, \beta \in \mathbb{N}^n\}$ in each of these cases.

The non-linear property (1.21) of the transition probabilities, which holds for Markov processes belonging to the special class B_1, determined the structure of a powerful analytic apparatus to study branching processes ([106], Foreword). The following problem arises in connection with the structure (1.23): find non-linear properties of transition probabilities for other special classes.

CHAPTER II

APPLICATIONS IN FORMAL KINETICS

Markov processes with interaction, regarded as particular cases of Markov processes on $\mathbb{N}^n$, have been defined in many papers devoted to specific problems of physical kinetics, chemical kinetics, population dynamics in ecological systems, queueing theory, and so on; this is due to the intuitiveness of the state space $\mathbb{N}^n$.

In the present chapter and the next we give a survey of results for diverse schemes of interactions.

In the structure (1.23) the choice of infinitesimal characteristics for the classes B_1, B_2, and B_3 of Markov processes is determined by phenomenological laws of kinetics. Branching processes of class B_1 describe the exponential growth of the number of "active" particles at the initial stage of a chain reaction [22], whereas processes of class B_3 correspond to power-law growth of the number of "active" particles in a reaction of the same type [124]. In § 2.1.2 we present theorems on the asymptotic behaviour of the mean number of particles in birth-and-death processes of linear, power-law, and Poisson types. In § 2.2 we suggest models of chemical reactions in the form of processes of classes B_1 and B_2. A relationship between a model of a bimolecular chemical reaction in the form of a Markov process of class B_2 and the macroscopic description of the kinetics of this reaction in the form of the law of active mass was established in [79]. Processes belonging to M_1 can serve as models of quite different reactions.

Stochastic models with interaction of particles for discrete states coinciding with those studied in the paper have been investigated by numerical experiments (see, for instance, [94], [95], [109], where actual physical-chemical phenomena were studied). In § 2.3 we briefly present results of statistical simulation of a branching process with a 'predator-prey' interaction scheme.

§ 2.1. Types of birth-and-death processes

Let the transition probabilities $P_{ij}(t) = \mathsf{P}\{\xi_t = j \mid \xi_0 = i\}$ of a homogeneous Markov process ξ_t on the state space $\mathbb{N} = \{0, 1, 2, \dots\}$, where $t \in [0, \infty)$, be representable as $t \to 0+$ in the form

$$P_{i,i-1}(t) = \varphi_i p_0 t + o(t), \quad P_{ii}(t) = 1 - \varphi_i t + o(t), \quad P_{i,i+1}(t) = \varphi_i p_2 t + o(t), \tag{2.1}$$

where $p_0 \geqslant 0$, $p_2 \geqslant 0$, $p_0 + p_2 = 1$, $\varphi_0 = 0$, and $\varphi_i > 0$ for $i = 1, 2, \dots$. The process stays in the initial state i during a random time τ_i with $\mathsf{P}\{\tau_i \leqslant t\} = 1 - e^{-\varphi_i t}$, after which it passes either to the state $i - 1$ with probability p_0 or to the state $i + 1$ with probability p_2. The further evolution of the process is similar, and the state 0 is absorbing. An 'embedded Markov chain' for such a process is a random walk on $\mathbb{N}$ with absorbing boundary point 0.

A birth-and-death process ξ_t is a process belonging to M_1 and having the interaction complex $\varepsilon = 1$. If a process in M_1 has only a single interaction complex, then [48] we can represent the first and second system of equations for the transition probabilities in a concise form by using generating functions. We can then apply the operator of generalized differentiation D_z and use an eigenfunction $e(z)$ of this operator [29]:

$$D_z\left(\sum_{j=0}^{\infty} a_j z^j\right) = \sum_{j=1}^{\infty} a_j \varphi_j z^{j-1}, \qquad e(z) = \sum_{i=0}^{\infty} \frac{z^i}{\varphi_1 \cdots \varphi_i}. \tag{2.2}$$

Let us introduce the following generating functions:

$$G_j(t; z) = \sum_{i=0}^{\infty} \frac{z^i}{\varphi_1 \cdots \varphi_i} P_{ij}(t), \ \ j \in \mathbb{N}; \quad F_i(t; s) = \sum_{j=0}^{\infty} P_{ij}(t) s^j, \ \ i \in \mathbb{N}; \quad |s| < 1.$$

Proposition 2.1. *The first and second Kolmogorov systems of differential equations for a birth-and-death process ξ_t can be represented in the form*

$$\frac{\partial G_j(t;z)}{\partial t} = z(p_2 D_z^2 + p_0 - D_z)G_j(t;z), \quad G_j(0;z) = \frac{z^j}{\varphi_1 \cdots \varphi_j}, \tag{2.3}$$

$$\frac{\partial F_i(t,s)}{\partial t} = (p_2 s^2 + p_0 - s)D_s(F_i(t,s)), \quad F_i(0,s) = s^i. \tag{2.4}$$

Proof. The first system of differential equations (1.2) for the process ξ_t becomes

$$\frac{dP_{0j}(t)}{dt} = -\varphi_0 P_{0j}(t),$$
$$\frac{dP_{ij}(t)}{dt} = p_0\varphi_i P_{i-1,j}(t) - \varphi_i P_{ij}(t) + p_2\varphi_i P_{i+1,j}(t), \quad i = 1, 2, \ldots,$$

with the initial conditions $P_{ii}(0) = 1$ and $P_{ij}(0) = 0$ for $i \neq j$. Using the definition of the function $G_j(t;z)$, we obtain the equation (2.3):

$$\begin{aligned}
\frac{\partial G_j}{\partial t} &= \sum_{i=0}^{\infty} \frac{z^i}{\varphi_1 \cdots \varphi_i} \frac{dP_{ij}(t)}{dt} \\
&= zp_2 \sum_{i=1}^{\infty} \frac{z^{i-1}}{\varphi_1 \cdots \varphi_{i-1}} P_{i+1,j}(t) \\
&\quad - z \sum_{i=1}^{\infty} \frac{z^{i-1}}{\varphi_1 \cdots \varphi_{i-1}} P_{ij}(t) + zp_0 \sum_{i=1}^{\infty} \frac{z^{i-1}}{\varphi_1 \cdots \varphi_{i-1}} P_{i-1,j}(t) \\
&= zp_2 D_z^2(G_j) - zD_z(G_j) + zp_0 G_j = z(p_2 D_z^2 - D_z + p_0)G_j.
\end{aligned}$$

The equation (2.4) is a particular case of the equation (1.11). This proves Proposition 2.1.

Let us introduce the double generating function

$$\mathcal{F}(t;z;s) = \sum_{i=0}^{\infty} \frac{z^i}{\varphi_1 \cdots \varphi_i} F_i(t;s) = \sum_{i,j=0}^{\infty} \frac{z^i}{\varphi_1 \cdots \varphi_i} P_{ij}(t) s^j = \sum_{j=0}^{\infty} G_j(t;z) s^j. \tag{2.5}$$

Corollary 2.2. *The first and second systems of equations for the process ξ_t can be represented in the form*

$$\frac{\partial \mathcal{F}}{\partial t} = z(p_2 D_z^2 + p_0 - D_z)\mathcal{F}, \tag{2.6}$$

$$\frac{\partial \mathcal{F}}{\partial t} = (p_2 s^2 + p_0 - s)D_s(\mathcal{F}), \quad \mathcal{F}(0;z;s) = e(zs). \tag{2.7}$$

The types of Markov processes are defined in dependence on the function $\varphi_i = \varphi(i)$ [104]. For birth-and-death processes of *Poisson* type we set $\varphi_0 = 0$ and $\varphi_i = \lambda$ for $i = 1, 2, \ldots$ $(\lambda > 0)$; then $D_z(f) = \lambda(f(z) - f(0))/z$ and $e(z) = \lambda/(\lambda - z)$.

For processes of *linear* type we set $\varphi_i = i\lambda + \lambda_1$; such a process is of class B_1. If $\varphi_i = i\lambda$, then $D_z = \lambda\,(d/dz)$ and $e(z) = e^z$. For a process of *quadratic* type [38] we set $\varphi_i = i^2\lambda + i\lambda_1 + \lambda_2$. If $\varphi_i = i(i-1)\lambda$, then $D_z = \lambda z\,(d^2/dz^2)$, and the process is of class B_2. For a process of *polynomial* type [82] we set $\varphi_i = i^k\lambda + i^{k-1}\lambda_1 + \cdots + \lambda_k$; in particular, processes of *cubic* and *biquadratic* types have been studied [81]. For a process of *power-law* type we have $\varphi_i = i^\rho\lambda$ with $0 < \rho < 1$.

The Kolmogorov equations for the above birth-and-death processes have a unique solution if either $p_0 \geqslant p_2$ or $p_0 < p_2$ and $\sum_{i=1}^{\infty} \varphi_i^{-1} = \infty$ ([30], Chap. 7, § 4).

2.1.1. Exact solutions of the Kolmogorov equations. There are few cases in which an explicit solution of the first and second systems of differential equations for Markov processes with countable state space can be found; the known solutions relate to the above birth-and-death processes and some of their modifications.

Let $\delta_i^i = 1$ and $\delta_j^i = 0$ for $i \neq j$.

Simple death process. By setting $p_0 = 1$ in the definition (2.1), we obtain a simple death process; the second equation becomes

$$\frac{\partial F_i(t;s)}{\partial t} = (1-s)D_s(F_i(t;s)), \qquad F_i(0;s) = s^i.$$

For a death process the expressions for the transition probabilities (provided that the numbers φ_i are pairwise distinct) are as follows: $P_{0j}(t) = \delta_j^0$ for $j \in \mathbb{N}$, $P_{ij}(t) = 0$ for $j > i \geqslant 1$, and

$$P_{ij}(t) = \varphi_{j+1}\cdots\varphi_i \sum_{n=j}^{i} \frac{e^{-\varphi_n t}}{(\varphi_i - \varphi_n)\cdots(\varphi_{n+1} - \varphi_n)(\varphi_{n-1} - \varphi_n)\cdots(\varphi_j - \varphi_n)} \tag{2.8}$$

for $j \leqslant i$.

Pure birth process. By setting $p_2 = 1$ in the definition (2.1), we obtain the second equation for the birth process,

$$\frac{\partial F_i(t;s)}{\partial t} = (s^2 - s)D_s(F_i(t;s)), \qquad F_i(0;s) = s^i.$$

The expressions for the transition probabilities (provided that the numbers φ_i are pairwise distinct) are as follows [30]: $P_{0j}(t) = \delta_j^0$ for $j \in \mathbb{N}$, $P_{ij}(t) = 0$ for $j < i$, and

$$P_{ij}(t) = \varphi_i\cdots\varphi_{j-1} \sum_{n=i}^{j} \frac{e^{-\varphi_n t}}{(\varphi_i - \varphi_n)\cdots(\varphi_{n-1} - \varphi_n)(\varphi_{n+1} - \varphi_n)\cdots(\varphi_j - \varphi_n)} \tag{2.9}$$

for $j \geqslant i \geqslant 1$.

Birth-and-death process of Poisson type. The second equation is

$$\frac{\partial F_i(t,s)}{\partial t} = \lambda(p_2 s^2 + p_0 - s)\frac{F_i(t;s) - F_i(t;0)}{s}, \qquad F_i(0,s) = s^i.$$

The solution of this equation leads to the following formulae for the transition probabilities (for $0 < p_0 < 1$; see [27], [93]): $P_{0j}(t) = \delta_j^0$ for $j \in \mathbb{N}$ and

$$\begin{aligned} P_{i0}(t) &= i \int_0^t e^{-\lambda x} \left(\frac{p_0}{p_2}\right)^{i/2} I_i(2\sqrt{p_0 p_2}\,\lambda x)\,dx, \quad i = 1, 2, \dots, \\ P_{ij}(t) &= e^{-\lambda t} \left(\frac{p_0}{p_2}\right)^{(i-j)/2} \big(I_{i-j}(2\sqrt{p_0 p_2}\,\lambda t) - I_{i+j}(2\sqrt{p_0 p_2}\,\lambda t)\big), \quad i, j \neq 0, \end{aligned} \tag{2.10}$$

where $I_i(t)$ are the modified Bessel functions.

Birth-and-death process of linear type. Let $D_s = \lambda\,(d/ds)$ in the equation (2.4). For $p_0 \neq p_2$, the solution of the first-order partial differential equation is ([71],[106])

$$F_i(t; s) = \left(\frac{p_0(1 - e^{(p_0-p_2)\lambda t}) - s(p_0 - p_2 e^{(p_0-p_2)\lambda t})}{p_2 - p_0 e^{(p_0-p_2)\lambda t} - s p_2 (1 - e^{(p_0-p_2)\lambda t})}\right)^i, \quad i \in \mathbb{N}. \tag{2.11}$$

Explicit solutions of the Kolmogorov equation for $p_0 = p_2$ and for some other birth-and-death processes of linear type are presented in [106]. Processes of linear type have been studied in [75], [37], and elsewhere.

2.1.2. Probabilistic models of chain reactions. In nuclear physics, the initial stage of a chain reaction is described [106] by the amount $x(t)$ of 'active' matter at any time $t \in [0, \infty)$; a detailed description of physical processes of this kind can be found in [22], [33], [119], [21]. The amount $x(t)$ of matter can grow exponentially, according to a power law, or linearly. When describing the growth of the active matter in a chain reaction, the equation of formal kinetics is applied [23],

$$\dot{x} = \lambda x^\rho, \qquad x(0) = x_0, \qquad 0 \leqslant \rho \leqslant 1, \tag{2.12}$$

where the coefficient $\lambda > 0$ is called the *reaction rate constant*. If $\rho = 1$ in the equation (2.12), then $x(t) = x_0 e^{\lambda t}$. For $0 < \rho < 1$ the solution is

$$x(t) = \left[(1-\rho)\left(\frac{x_0^{1-\rho}}{1-\rho} + \lambda t\right)\right]^{1/(1-\rho)}; \tag{2.13}$$

hence, $x(t) \sim C_0(\lambda t)^{1/(1-\rho)}$ as $t \to \infty$, where $C_0 > 0$. If $\rho = 0$ in (2.12), then $x(t) = x_0 + \lambda t$.

Corresponding to the deterministic model (2.12) is the probabilistic model of a chain reaction in the form of a Markov birth-and-death process ξ_t, namely, the process belonging to the set M_1 is determined by the triple $\varepsilon = 1$, $\{p_0, p_2 \geqslant 0$, $p_0 + p_2 = 1\}$, $\{\varphi_0 = 0$, $\varphi_i > 0$, $i = 1, 2, \dots\}$. When interpreting an event $\{\xi_t = i\}$ as the presence of i particles of type T, we obtain the scheme $T \to kT$, $k = 0, 2$, in which either one particle of active type T disappears with probability p_0 or two new particles of type T appear with probability p_2. We set $h(s) = p_0 + p_2 s^2$. The probabilistic model is characterized by the mean number $A_i(t) = \mathsf{E}(\xi_t \mid \xi_0 = i)$ of particles at any time t.

Model of a chain reaction with exponential growth. For a birth-and-death process of linear type in which $\varphi_i = \lambda i$, $i \in \mathbb{N}$ ($\lambda > 0$), we have the following second equation for the generating function of the transition probabilities:

$$\frac{\partial F_i(t;s)}{\partial t} = \lambda(h(s) - s)\frac{\partial F_i(t;s)}{\partial s}, \qquad F_i(0,s) = s^i. \tag{2.14}$$

By differentiating with respect to s, we obtain from (2.14) the following equation for the expectation of the number $A_i(t) = (\partial F_i(t;s)/\partial s)|_{s=1}$ of particles:

$$\frac{dA_i}{dt} = aA_i, \qquad A_i(0) = i$$

(where $a = \lambda(h'(1) - 1)$), which coincides with the equation (2.12) for $\rho = 1$.

Theorem 2.3 ([106], [33]). *For a birth-and-death process of linear type,*

$$A_i(t) = ie^{at}. \tag{2.15}$$

The number a is called the *criticality parameter*. If $a > 0$, then the mean number of active particles grows exponentially. In the theory of branching processes the general scheme $T \to kT$ studied has one particle turning into k new particles with probability distribution $\{p_k\}$. Then $h(s) = \sum_{k=0}^{\infty} p_k s^k$ in the equation (2.14), and the formula (2.15) also holds. In the table we present the probabilities p_k for the occurrence of k secondary neutrons in nuclear fission of uranium ^{235}U by thermal neutrons [22].

k	0	1	2	3	4	5	6	7
p_k	0.0333	0.1745	0.3349	0.3028	0.1231	0.0281	0.0032	0.0001

Model of a chain reaction with power-law growth. The probabilistic model of a chain reaction corresponding to the law (2.13) is a birth-and-death process in the class B_3. Let $U(x)$ be the distribution function of a non-negative random variable and also the distribution function of a stable law with parameter ρ, $0 < \rho < 1$. The Laplace transform of the distribution $U(x)$ ([26], vol. 2, Chap. 13, § 6) is

$$\omega(p) = \int_0^\infty e^{-px}\, U\{dx\} = e^{-\psi(p)}, \qquad p \geqslant 0,$$

where $\psi(p) = \lambda p^\rho$ and $\lambda > 0$. According to the definition in § 1.6, we obtain $\varphi_i = \lambda i^\rho$ for $i \in \mathbb{N}$.

Theorem 2.4 [125]. *Let the criticality parameter and the exponent of a birth-and-death process of power-law type satisfy the relations $a > 0$ and $1/2 < \rho < 1$. Then as $t \to \infty$*

$$A_i(t) \sim C_i(at)^{1/(1-\rho)}, \qquad C_i > 0.$$

Similar results were obtained in [14].

Model of a chain reaction with linear growth. For the case in which $\varphi_0 = 0$ and $\varphi_i = \lambda$ for $i = 1, 2, \dots$, the next assertion was obtained in [93] on the basis of explicit expressions (2.10) for the transition probabilities.

Theorem 2.5 [93]. *Let the criticality parameter a be positive for a birth-and-death process of Poisson type. Then as $t \to \infty$*

$$A_i(t) \sim C_i a t, \qquad C_i > 0.$$

The above laws for the growth of the mean number of active particles in chain reactions are shown in Fig. 1.

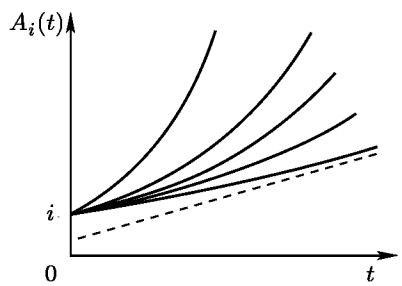

Figure 1. Means for schemes of the form $T \to kT$

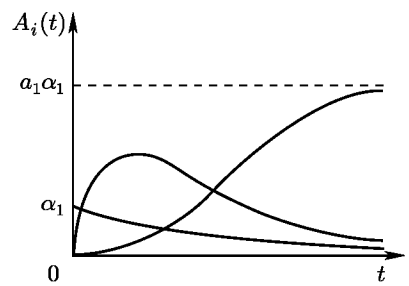

Figure 2. Means for schemes of the form $T_1 \to T_2 \to T_3$

§ 2.2. Markov models of chemical reactions

The general scheme of a chemical reaction with the reagents $T_1, \dots, T_n$ can be represented in the form (1.9) for fixed vectors $\gamma^1, \dots, \gamma^l$ [23]. A single row in the scheme (1.9) corresponds to an elementary act of reaction. The kinetics of the chemical reaction is described by the amount $x_i(t)$ of any reagent T_i at any time $t \in [0, \infty)$, where $i = 1, \dots, n$. The functions $x_1(t), \dots, x_n(t)$ satisfy a system of differential equations of the form

$$\begin{cases} \dot{x}_1 = f_1(x_1, \dots, x_n), \\ \quad \dots \\ \dot{x}_n = f_n(x_1, \dots, x_n), \end{cases}$$

with initial conditions $x_1(0) = x_1^0, \dots, x_n(0) = x_n^0$. The form of the functions $f_1, \dots, f_n$ is determined according to the laws of formal chemical kinetics [23].

Probabilistic models of chemical reactions in the form of Markov processes on the state space $\mathbb{N}^n$ were introduced in [79]. Diverse examples of such Markov processes are given in [10] (Chap. 8, Applications to Chemistry) and [89]. The branching process with the scheme (1.9) and with fixed vectors $\gamma^1, \dots, \gamma^l$ was defined in [92] and called a generalized birth-and-death process. For a detailed description of the physical assumptions under which a representation of a chemical reaction in the form of a Markov process is admissible, see [121], §§ 7.1, 7.2. Below we present the main equations of chemical kinetics and the corresponding Markov processes on $\mathbb{N}^n$.

We note that the interaction scheme (1.9) can take into account both the formation of the final product in a chemical reaction (processes with particles of final types) and the introduction of reagents into the system from outside (processes with immigration of particles).

2.2.1. Monomolecular reactions. Chemical reactions in which only one particle takes part in any elementary act are said to be monomolecular; the scheme (1.9) becomes

$$\begin{cases} T_1 \to \gamma_1^1 T_1 + \gamma_2^1 T_2 + \dots + \gamma_n^1 T_n, \\ \quad \dots \\ T_l \to \gamma_1^l T_1 + \gamma_2^l T_2 + \dots + \gamma_n^l T_n. \end{cases} \tag{2.16}$$

Models of monomolecular chemical reactions are Markov processes of class B_1 [105]. In Chap. 4, § 6 of [106] a classification of types of particles in branching processes is given which is a classification of schemes of the form (2.16), in Chap. 4, § 4 general methods for computing the mean number of particles of type T_i, $i = 1, \dots, n$, are presented, and in Chap. 4, § 7 asymptotic properties of the means in indecomposable branching processes are investigated as $t \to \infty$ and the diversity of the asymptotic behaviour of the means for decomposable processes is indicated. For chemical reactions, schemes of the form (2.16) correspond to decomposable processes as a rule; the following examples relate to such processes.

Reaction $T_1 \to T_2$. The reaction is described by the amount $x_1(t)$ of reagent T_1 and the amount $x_2(t)$ of reagent T_2. The following law of kinetics is assumed to hold [23]:

$$\dot{x}_1 = -\lambda x_1, \qquad \dot{x}_2 = \lambda x_1, \tag{2.17}$$

where $\lambda > 0$ is the reaction rate constant.

Corresponding to the deterministic model (2.17) is a process of class B_1 with transmutation of particles for two types of particles T_1 and T_2 and one interaction complex ([10], [89]), namely, the Markov process $\xi(t) = (\xi_1(t), \xi_2(t))$, $t \in [0, \infty)$, (on the state space $\mathbb{N}^2$) determined by $\varepsilon = (1, 0)$, $\{p_{(0,1)} = 1\}$, $\{\varphi_\alpha = \lambda\alpha_1$, $\alpha \in \mathbb{N}^2$, $\lambda > 0\}$. The second Kolmogorov equation for the generating function $F_{(\alpha_1,\alpha_2)}(t; s_1, s_2)$ of the transition probabilities becomes

$$\frac{\partial F_\alpha(t; s)}{\partial t} = \lambda(s_2 - s_1)\frac{\partial F_\alpha(t; s)}{\partial s_1}, \tag{2.18}$$

and for the initial condition we take $F_\alpha(0; s) = s_1^{\alpha_1}$.

Differentiating (2.18) with respect to s_1 or s_2, we obtain the following system of equations for the mean numbers $A_1(t) = \mathsf{E}\xi_1(t)$ and $A_2(t) = \mathsf{E}\xi_2(t)$ of particles:

$$\frac{dA_1}{dt} = -\lambda A_1, \qquad \frac{dA_2}{dt} = \lambda A_1,$$

with the initial conditions $A_1(0) = \alpha_1$ and $A_2(0) = 0$. Hence, $A_1(t) = \alpha_1 e^{-\lambda t}$ and $A_2(t) = \alpha_1(1 - e^{-\lambda t})$. The solution of (2.18) is

$$F_{(\alpha_1,0)}(t; s_1, s_2) = \left(s_1 e^{-\lambda t} + s_2(1 - e^{-\lambda t})\right)^{\alpha_1}. \tag{2.19}$$

It follows from (2.19) by the formula (1.5) that $D_1(t) = D_2(t) = \alpha_1 e^{-\lambda t}(1 - e^{-\lambda t})$, where $D_1(t) = \mathsf{D}\xi_1(t)$ and $D_2(t) = \mathsf{D}\xi_2(t)$ are the variances of the numbers of particles.

The generating function (2.19) corresponds to the binomial distribution on the states $\{(\alpha_1, 0), (\alpha_1 - 1, 1), \dots, (0, \alpha_1)\}$. The de Moivre–Laplace theorem on the approximation of a binomial distribution by a normal distribution acquires the following form.

Theorem 2.6. *Let $\xi_i(t)$ be the number of particles of type T_i at time t for a branching process with the scheme $T_1 \to T_2$ and let there be α_1 particles of type T_1 at time $t = 0$. Then for any $t > 0$*

$$\lim_{\alpha_1\to\infty} \mathsf{P}\left\{ \frac{\xi_i(t) - A_i(t)}{\sqrt{D_i(t)}} \leqslant x \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du, \qquad i = 1, 2. \tag{2.20}$$

Observing a chemical reaction, one obtains a set of experimental data, and the results of the statistical data processing give approximations for the kinetic curves $x_1(t)$ and $x_2(t)$. As a rule, it is assumed that the experimental points deviate from the kinetic curve according to a normal law. The limit Theorem 2.6 shows that the deviation of the 'experimental' values $\xi_i(t)$ from the values $A_i(t)$ is indeed distributed according to a normal law under the assumptions of the probabilistic model under consideration.

Successive reactions $T_1 \to T_2 \to T_3$. In [97] a decomposable branching process was considered with three types of particles T_1, T_2, T_3 and the following scheme of transmutation of particles:

$$\begin{cases} T_1 \to \gamma_1^1 T_1 + \gamma_2^1 T_2 + \gamma_3^1 T_3, \\ T_2 \to \gamma_2^2 T_2 + \gamma_3^2 T_3. \end{cases} \tag{2.21}$$

The generating function $F_{(\alpha_1,\alpha_2,\alpha_3)}(t; s_1, s_2, s_3)$ of the transition probabilities satisfies the second equation (where $\lambda > 0$ and $\mu > 0$),

$$\frac{\partial F_\alpha(t;s)}{\partial t} = \lambda(h_1(s_1, s_2, s_3) - s_1)\frac{\partial F_\alpha(t;s)}{\partial s_1} + \mu(h_2(s_2, s_3) - s_2)\frac{\partial F_\alpha(t;s)}{\partial s_2}, \tag{2.22}$$

and we take the initial condition in the form $F_\alpha(0; s) = s_1^{\alpha_1}$. Here $h_1(s_1, s_2, s_3)$ and $h_2(s_2, s_3)$ stand for the probability generating functions. We set

$$a_j^i = \left.\frac{\partial h_i}{\partial s_j}\right|_{s=1} - \delta_j^i, \quad b_{jk}^i = \left.\frac{\partial^2 h_i}{\partial s_j \partial s_k}\right|_{s=1}, \qquad i, j, k = 1, 2, 3.$$

By using the methods of [106] (Chap. 4), explicit expressions for the mean number $A_i(t)$ of particles of type T_i, $i = 1, 2, 3$, were obtained from the equation (2.22) in [97]. In Fig. 2 we present the shape of $A_i(t)$ for $a_1^1 < 0$, $a_2^1 > 0$, $a_2^2 < 0$, and $a_3^2 > 0$. This shape of the kinetic curves is typical for monomolecular chemical reactions $T_1 \to T_2 \to T_3$ ([23], Fig. 56).

Theorem 2.7 [97]. *Suppose that $a_1^1 < 0$, $a_2^1 > 0$, $a_2^2 < 0$, $a_3^2 > 0$, $b_{11}^1 > 0$ or $b_{22}^2 > 0$, and $h_2(0, 1) > 0$ for a branching process with the scheme* (2.21). *Let there be α_1 particles of type T_1 at time $t = 0$, and let η_{α_1} be the number of final particles of type T_3 generated by the original particles. Then*

$$\lim_{\alpha_1\to\infty} \mathsf{P}\left\{ \frac{\eta_{\alpha_1} - a_1\alpha_1}{\sigma_1\sqrt{\alpha_1}} \leqslant x \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du, \tag{2.23}$$

where

$$a_1 = \frac{a_2^1 a_3^2 - a_3^1 a_2^2}{a_1^1 a_2^2}, \quad \sigma_1^2 = b_1 + a_1 - (a_1)^2,$$
$$b_1 = -\frac{a_2^1 b_2 + b_{11}^1 (a_1)^2 + 2b_{12}^1 a_1 a_2 + b_{22}^1 (a_2)^2 + 2b_{13}^1 a_1 + 2b_{23}^1 a_2 + b_{33}^1}{a_1^1},$$
$$a_2 = -\frac{a_3^2}{a_2^2}, \quad b_2 = -\frac{b_{22}^2 (a_2)^2 + 2b_{23}^2 a_2 + b_{33}^2}{a_2^2}.$$

Theorem 2.7 generalizes Theorem 1 in [106], Chap. 5, § 5, where a branching process with the scheme $T_1 \to \gamma_1 T_1 + \gamma_2 T_2$ was considered. A process with the general scheme $T_1 \to T_2 \to \cdots \to T_n$ was studied in [98]; for other decomposable branching processes, see the survey [122].

For other schemes (2.16), assertions of the form (2.20) and (2.23) follow from the branching property (1.21), which means for a fixed $t > 0$ that the number of particles of type T_i is a sum of independent random variables; to derive limit theorems one can apply the method of characteristic functions to the relation (1.21). In [106] (Chap. 5, § 5) it was established that, along with the normal distribution, there are other limit distributions for large initial numbers of particles.

2.2.2. Bimolecular reactions. A chemical reaction whose elementary act involves two particles is said to be bimolecular. Let $x_1(t)$, $x_2(t)$, $x_3(t)$ be the amounts of the reagents T_1, T_2, T_3 at time t for a bimolecular reaction with the scheme $T_1 + T_2 \to T_3$. In formal kinetics the following law of active mass is assumed [23]:

$$\dot{x}_1 = -\lambda x_1 x_2, \quad \dot{x}_2 = -\lambda x_1 x_2, \quad \dot{x}_3 = \lambda x_1 x_2. \tag{2.24}$$

For a probabilistic model of the reaction we consider a process of class B_2 with three types of particles T_1, T_2, T_3 and one interaction complex ([10], [89]), namely, a Markov process $\xi(t) = (\xi_1(t), \xi_2(t), \xi_3(t))$, $t \in [0, \infty)$, (on the state space $\mathbb{N}^3$) determined by $\varepsilon = (1, 1, 0)$, $\{p_{(0,0,1)} = 1\}$, $\{\varphi_\alpha = \lambda \alpha_1 \alpha_2,\ \alpha \in \mathbb{N}^3,\ \lambda > 0\}$. The second equation for the generating function $F_{(\alpha_1,\alpha_2,\alpha_3)}(t; s_1, s_2, s_3)$ of the transition probabilities is

$$\frac{\partial F_\alpha(t; s)}{\partial t} = \lambda (s_3 - s_1 s_2) \frac{\partial^2 F_\alpha(t; s)}{\partial s_1 \partial s_2}, \qquad F_\alpha(0; s) = s^\alpha.$$

We note that $\xi(t)$ is a death process on the states $\{(\alpha_1, \alpha_2, \alpha_3),\ (\alpha_1 - 1, \alpha_2 - 1, \alpha_3 + 1),\ \ldots,\ (\alpha_1 - \alpha_2, 0, \alpha_3 + \alpha_2)\}$ with the absorbing state $(\alpha_1 - \alpha_2, 0, \alpha_3 + \alpha_2)$ (under the assumption that $\alpha_1 \geqslant \alpha_2$). The explicit expressions for the transition probabilities of the process $\xi(t)$ are given by the formula (2.8) for the corresponding function φ.

In the probabilistic model under consideration, the mean numbers $A_i(t) = (\partial F_\alpha(t; s)/\partial s_i)|_{s=1}$ of particles of the type T_i, $i = 1, 2, 3$, do not satisfy the equations (2.24). These equations hold approximately for the means $A_i(t)$ provided that the initial number of particles is large; one sets $\alpha = (n\alpha_1, n\alpha_2, n\alpha_3)$ and passes to

the limit as $n \to \infty$ (this passage corresponds to the thermodynamical limit used in statistical physics [102]). Conditions for this deterministic approximation of a process $\xi(t)$ were studied in [10] (§ 8.2, Sect. C) on the basis of explicit expressions for the transition probabilities.

From the point of view of the above passage to the limit, a Markov process of class B_2 on $\mathbb{N}^n$ with the interaction scheme $T_i + T_j \to T_k + T_l$, $i, j, k, l = 1, \dots, n$, was studied in [79]. The equations of formal kinetics for other bimolecular schemes and the corresponding Markov processes of class B_2 are presented in § 2.3 and in §§ 3.3.2, 4.2.1 (see also [32], [57]).

Class B_4. When studying chemical reactions, we always restrict ourselves to interactions involving at most two particles, that is, in the scheme (1.9) we assume that $|\varepsilon^i| \leqslant 2$ for $i = 1, \dots, l$. In formal kinetics one considers a generalization of the law (2.24); for the bimolecular reaction $T_1 + T_2 \to T_3$,

$$\dot{x}_1 = -\lambda x_1^{\rho_1} x_2^{\rho_2}, \quad \dot{x}_2 = -\lambda x_1^{\rho_1} x_2^{\rho_2}, \quad \dot{x}_3 = \lambda x_1^{\rho_1} x_2^{\rho_2}, \qquad \rho_1 > 0, \quad \rho_2 > 0$$

[23] (cf. (2.12)). On the state space $\mathbb{N}^n$ one can introduce the class B_4 of Markov processes (by analogy with the definition of the class B_3) that corresponds to the equations of the above form. The structure (1.23) is then supplemented by the relations $M_1 \supset B_4 \supset B_2$, $M_1 \supset B_4 \supset B_3$.

§ 2.3. 'Predator-prey' stochastic process

The dynamics of an ecological predator-prey' system is described by the number $x_1(t)$ of 'predators' and the number $x_2(t)$ of 'preys' at any time t. The following system of non-linear differential equations is used [123]:

$$\dot{x}_1 = -\mu x_1 + \lambda x_1 x_2, \qquad \dot{x}_2 = \rho x_2 - \lambda x_1 x_2, \tag{2.25}$$

where λ, μ, and ρ are some positive constants.

Corresponding to the deterministic model is a Markov process $\xi(t) = (\xi_1(t), \xi_2(t))$, $t \in [0; \infty)$, on the state space $\mathbb{N}^2$ which is a branching process with two types of particles T_1 and T_2 and the interaction scheme [20]

$$\begin{cases} T_1 + T_2 \to 2T_1, \\ T_1 \to 0, \\ T_2 \to 2T_2. \end{cases} \tag{2.26}$$

The second Kolmogorov equation for the generating function $F_{(\alpha_1,\alpha_2)}(t; s_1, s_2)$ of the transition probabilities is of the form

$$\frac{\partial F_\alpha(t;s)}{\partial t} = \lambda(s_1^2 - s_1 s_2)\frac{\partial^2 F_\alpha(t;s)}{\partial s_1 \partial s_2} + \mu(1 - s_1)\frac{\partial F_\alpha(t;s)}{\partial s_1} + \rho(s_2^2 - s_2)\frac{\partial F_\alpha(t;s)}{\partial s_2} \tag{2.27}$$

with the initial condition $F_\alpha(0; s) = s^\alpha$. An event $\{\xi(t) = (\beta_1, \beta_2)\}$ is interpreted as the existence of an aggregate of β_1 particles of type T_1 and β_2 particles of type T_2. The particles of type T_1 are 'predators', and the particles of type T_2 are 'preys';

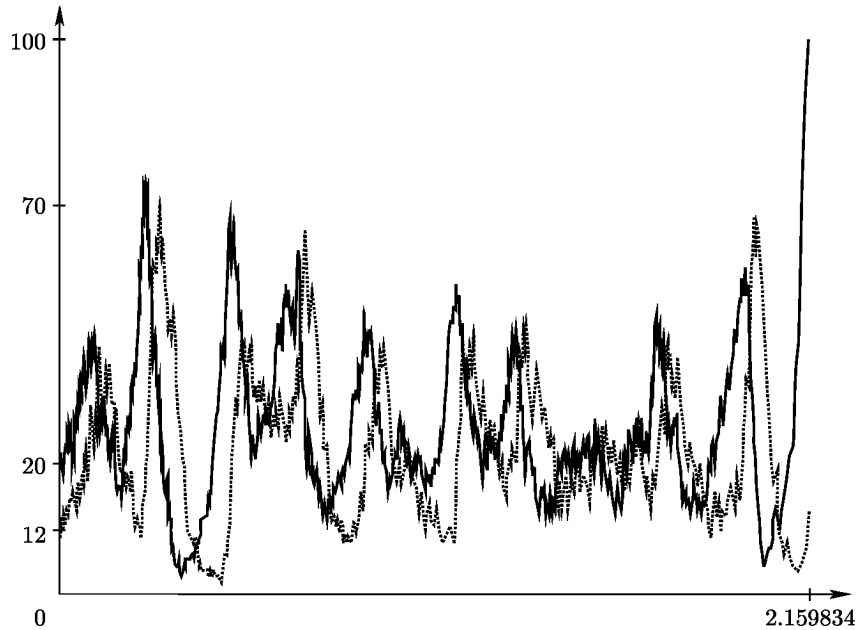

Figure 3. Realization of a predator-prey Markov process

the scheme (2.26) assumes that an interaction of a predator and a prey leads to an increase in the number of predators, predator-particles can die, and prey-particles produce others of the same type. This description of a predator-prey system agrees with the system of differential equations (2.25) if the initial number of particles is large; an approximation of the system (2.25) is obtained from the equation (2.27) by differentiating with respect to s_1 or s_2 and passing to the mean number of particles $A_i(t) = (\partial F_\alpha(t;s)/\partial s_i)|_{s=1}$, $i = 1, 2$.

In [116] an equivalent description of a Markov process $\xi(t)$ with the help of the random time $\tau_{(\beta_1,\beta_2)}$ during which the process stays in the state (β_1, β_2) in accordance with 1.3.1 was used in a computer-aided statistical simulation of the process. Statistical experiments enable one to make conjectures about the domain K of values of the parameters μ and ρ (one can set $\lambda = 1$ without loss of generality) in which the realizations of the process $\xi(t)$ stably have an oscillatory form (which is typical for deterministic predator-prey systems [123]): $K = \{\mu > 0, \rho > 0 : c_1 < \mu/\rho < c_2\}$, where c_1 and c_2 are some constants. If the parameters are outside the domain K, then the process either quickly falls into one of the states $(\gamma_1, 0)$, $\gamma_1 = 1, 2, \ldots$, and degenerates into the absorbing state $(0, 0)$, or falls into one of the states $(0, \gamma_2)$, $\gamma_2 = 1, 2, \ldots$, and goes to infinity. In Fig. 3 we show an example of the realization of a process $(\xi_1(t), \xi_2(t))$, $t \in [0, 2.159834]$, for the initial conditions $\xi_1(0) = 12$, $\xi_2(0) = 20$ and for the values $\lambda = 1$, $\mu = 31$, and $\rho = 27$ of the parameters [116].

A survey of literature on computer-aided simulation of predator-prey systems can be found in [72]; see also [95].

CHAPTER III

STATIONARY AND FINAL PROBABILITIES

Let $P_{\alpha\beta}(t)$ be the transition probabilities of a homogeneous Markov process on the set $\mathbb{N}^n$ with continuous time $t \in [0, \infty)$. As was shown in the general theory of processes with countable state space [18], if the assumptions of § 1.1.1 are satisfied, then the limits

$$q_{\alpha\beta} = \lim_{t\to\infty} P_{\alpha\beta}(t) \tag{3.1}$$

exist for any $\alpha, \beta \in \mathbb{N}^n$; the numbers $q_{\alpha\beta}$ are called *limit probabilities*. The limit probabilities may not be a probability distribution, that is, $\sum_\beta q_{\alpha\beta} \leqslant 1$.

The limit behaviour of Markov processes can be of diverse nature; it is determined by the classification of the states of the process [18]. We can distinguish the principal cases in which a process falls into an absorbing state, goes to infinity, or tends to a stationary state. To find the final probabilities (the probabilities of degeneration of the process into an absorbing state), the stationary first Kolmogorov system of differential equations is solved; to find the limit stationary probability distribution, the stationary second Kolmogorov system of differential equations is solved. In § 3.1 the first and second stationary equations for the generating functions of the limit probabilities of Markov processes of class B_2 are obtained. In § 3.2 it is shown that the stationary distribution as $t \to \infty$ coincides with the canonical distribution used in equilibrium statistical physics (for certain processes of classes B_1 or B_2). A method proposed by the author for finding the final probabilities for processes with interaction by solving partial differential equations for the exponential (double) generating function is presented in § 3.3. We also give results on integral representations for the final probabilities for interaction schemes with particles of different types. Exact solutions of the Darboux–Picard boundary-value problem (Goursat problem) for second-order partial differential equations of hyperbolic type are obtained by the Riemann method.

§ 3.1. Equations for limit probabilities

Let a Markov process of class B_2 be determined by $\varepsilon^1, \{p^1_\gamma\}, \dots, \varepsilon^l, \{p^l_\gamma\}$. We introduce the multivariate generating functions for the probabilities (3.1):

$$g_\beta(z) = \sum_\alpha \frac{z^\alpha}{\alpha!} q_{\alpha\beta}, \quad f_\alpha(s) = \sum_\beta q_{\alpha\beta} s^\beta, \qquad \alpha, \beta \in \mathbb{N}^n.$$

The function $g_\beta(z)$ is an analytic function of the variables $z_1, \dots, z_n$, and the function $f_\alpha(s)$ is analytic in the domain $|s_1| < 1, \dots, |s_n| < 1$.

Theorem 3.1. *The exponential generating function $g_\beta(z)$ of the limit probabilities of a branching process with interaction satisfies the following linear partial differential equation for any $\beta \in \mathbb{N}^n$:*

$$\sum_{i=1}^{l} \lambda_i z^{\varepsilon^i} \left(h_i\left(\frac{\partial}{\partial z}\right) - \frac{\partial^{\varepsilon^i}}{\partial z^{\varepsilon^i}} \right) g_\beta(z) = 0. \tag{3.2}$$

Proof. Let us use the equation (1.14) obtained in Theorem 1.2 for the exponential generating function $G_\beta(t;z)$ of the transition probabilities to derive the stationary equation (3.2). As is known in the general theory of homogeneous Markov processes with countable state space [18], under the assumptions of § 1.1.1 we have

$$\lim_{t\to\infty} \frac{dP_{\alpha\beta}(t)}{dt} = 0, \qquad \alpha,\beta \in \mathbb{N}^n.$$

Hence, it follows from the uniform convergence of the series

$$\sum_\alpha \frac{z^\alpha}{\alpha!} \frac{dP_{\alpha\beta}(t)}{dt}$$

with respect to t (established in the proof of Theorem 1.2) that

$$\lim_{t\to\infty} \frac{\partial G_\beta(t;z)}{\partial t} = 0, \qquad \beta \in \mathbb{N}^n. \tag{3.3}$$

On the other hand, since the series

$$\sum_\alpha \frac{z^\alpha}{\alpha!} P_{\alpha\beta}(t)$$

is uniformly convergent in the domain $t \in [0,\infty)$, it follows that

$$g_\beta(z) = \lim_{t\to\infty} G_\beta(t;z), \qquad \beta \in \mathbb{N}^n. \tag{3.4}$$

The equation (1.14) together with the relations (3.3) and (3.4) implies (3.2). This proves Theorem 3.1.

Theorem 3.2. *The generating function $f_\alpha(s)$ of the limit probabilities of a branching process with interaction satisfies for $|s| \leqslant 1$ the following linear partial differential equation for any $\alpha \in \mathbb{N}^n$:*

$$\sum_{i=1}^{l} \lambda_i \big(h_i(s) - s^{\varepsilon^i}\big) \frac{\partial^{\varepsilon^i} f_\alpha(s)}{\partial s^{\varepsilon^i}} = 0. \tag{3.5}$$

Theorem 3.2 follows from Theorem 1.3.

§ 3.2. Stationary distribution for a system of interacting particles for discrete states

A stochastic system of particles (of n different types $T_1,\dots,T_n$) interacting with one another by complexes is considered. A state of the system is characterized by an n-dimensional vector $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n$, which means the existence of a group S_α of particles formed by α_1 particles of type T_1, …, α_n particles of type T_n; let $P_{\alpha\beta}(t)$, $\alpha,\beta \in \mathbb{N}^n$, be the probability of the event that the initial group S_α of particles passes to the group S_β of particles during the time interval $[0,t]$. Following § 1.3, we assume that the stochastic process has the Markov

property and is homogeneous with respect to time. The transition probability densities $a_{\alpha\beta} = (dP_{\alpha\beta}(t)/dt)|_{t=0+}$, $\alpha, \beta \in \mathbb{N}^n$, are determined as follows. We fix a set $A = \{\varepsilon^1, \dots, \varepsilon^l \in \mathbb{N}^n\}$ of vectors and a matrix $P = (p^i_j)^l_{i,j=1}$ whose entries satisfy the conditions $p^i_j \geqslant 0$ and $\sum_{j=1}^l p^i_j = 1$. Suppose that during the time Δt, $\Delta t \to 0$, the particles of any aggregate $S_{\varepsilon^i} = \varepsilon^i_1 T_1 + \dots + \varepsilon^i_n T_n$ interact with one another with probability $\lambda_i \Delta t + o(\Delta t)$, where $\lambda_i > 0$, and that an interaction of several groups of this kind during the time Δt can occur only with probability $o(\Delta t)$. As the result of this interaction, the complex S_{ε^i} of interacting particles is replaced with probability p^i_j by a new aggregate S_{ε^j} of particles, where $\sum_{i=1}^l p^i_j = 1$ and $p^i_i = 0$; the group $S_\alpha = \alpha_1 T_1 + \dots + \alpha_n T_n$ of particles passes into the group $S_{\alpha - \varepsilon^i + \varepsilon^j} = (\alpha_1 - \varepsilon^i_1 + \varepsilon^j_1) T_1 + \dots + (\alpha_n - \varepsilon^i_n + \varepsilon^j_n) T_n$ of particles. Then it follows from the above assumptions that

$$a_{\alpha\alpha} = -\sum_{i=1}^l \lambda_i \alpha_1^{[\varepsilon^i_1]} \cdots \alpha_n^{[\varepsilon^i_n]}, \quad a_{\alpha\beta} = \sum_{i,j:\, \beta = \alpha - \varepsilon^i + \varepsilon^j} \lambda_i \alpha_1^{[\varepsilon^i_1]} \cdots \alpha_n^{[\varepsilon^i_n]} p^i_j \ \ (\alpha \neq \beta). \tag{3.6}$$

For the Markov process under consideration (belonging to the class B_2) each of the possible vectors $\gamma^1, \dots, \gamma^l$ in the interaction scheme (1.9) belongs to the set $A = \{\varepsilon^1, \dots, \varepsilon^l\}$ of interaction complexes. The second Kolmogorov equation for the generating function (1.10) of the transition probabilities becomes

$$\frac{\partial F_\alpha(t;s)}{\partial t} = \sum_{i=1}^l \lambda_i (h_i(s) - s^{\varepsilon^i}) \frac{\partial^{\varepsilon^i} F_\alpha(t;s)}{\partial s^{\varepsilon^i}}, \qquad F_\alpha(0;s) = s^\alpha,$$

where $h_i(s) = \sum_{j=1}^l p^i_j s^{\varepsilon^j}$ for $i = 1, \dots, l$.

A state γ is said to be *attainable* from a state α, $\alpha \to \gamma$, if there is a $t_0 < \infty$ such that $P_{\alpha\gamma}(t_0) > 0$. Suppose that some power of the matrix P has only positive entries (the ergodicity condition [30]). Then $\varepsilon^i \to \varepsilon^j$ for any $i, j = 1, \dots, l$, and if $\alpha \to \gamma$, then $\gamma \to \alpha$ as well, that is, the set $K_\alpha = \{\gamma : \alpha \to \gamma\}$ forms a closed class of communicating states.

Theorem 3.3 [40]. *Let the matrix P satisfy the ergodicity condition and let there be an n-tuple $q = (q_1, \dots, q_n)$ of positive numbers such that*

$$\sum_{i=1}^l \lambda_i p^i_j q^{\varepsilon^i} - \lambda_j q^{\varepsilon^j} = 0, \qquad j = 1, \dots, l. \tag{3.7}$$

Then for any initial state α of the Markov process there is a limit stationary distribution $\{q_{\alpha\gamma},\ \gamma \in K_\alpha\}$ in the class K_α,

$$q_{\alpha\gamma} = \lim_{t\to\infty} P_{\alpha\gamma}(t), \qquad \alpha, \gamma \in K_\alpha,$$

and the generating function of the stationary probabilities is of the form

$$f_\alpha(s) = \sum_{\gamma \in K_\alpha} q_{\alpha\gamma} s^\gamma = \Big(\sum_{\gamma \in K_\alpha} \frac{q^\gamma}{\gamma!} \Big)^{-1} \Big(\sum_{\gamma \in K_\alpha} \frac{q^\gamma s^\gamma}{\gamma!} \Big). \tag{3.8}$$

Proof. Let the class K_α be finite. In a finite closed class of states of a Markov process with continuous time a limit distribution always exists, and it is unique [30]; to prove the theorem, it suffices to show that the generating function (3.8) satisfies the stationary second equation (3.5):

$$\begin{aligned}
&\sum_{i=1}^{l} \lambda_i (h_i(s) - s^{\varepsilon^i}) \frac{\partial^{\varepsilon^i}}{\partial s^{\varepsilon^i}} \biggl(\sum_{\gamma \in K_\alpha} \frac{q^\gamma s^\gamma}{\gamma!} \biggr) \\
&\quad = \sum_{i=1}^{l} \lambda_i \biggl(\sum_{j=1}^{l} p_j^i s^{\varepsilon^j} - s^{\varepsilon^i} \biggr) q^{\varepsilon^i} \sum_{\gamma \in K_\alpha} \frac{q^{\gamma - \varepsilon^i} s^{\gamma - \varepsilon^i}}{(\gamma - \varepsilon^i)!} \\
&\quad = \biggl(\sum_{j=1}^{l} \sum_{i=1}^{l} \lambda_i p_j^i s^{\varepsilon^j} q^{\varepsilon^i} - \sum_{i=1}^{l} \lambda_i s^{\varepsilon^i} q^{\varepsilon^i} \biggr) \biggl(\sum_{\gamma \in K_\alpha} \frac{q^{\gamma - \varepsilon^1} s^{\gamma - \varepsilon^1}}{(\gamma - \varepsilon^1)!} \biggr) \\
&\quad = \biggl(\sum_{\gamma \in K_\alpha} \frac{q^{\gamma - \varepsilon^1} s^{\gamma - \varepsilon^1}}{(\gamma - \varepsilon^1)!} \biggr) \sum_{j=1}^{l} s^{\varepsilon^j} \biggl(\sum_{i=1}^{l} \lambda_i p_j^i q^{\varepsilon^i} - \lambda_j q^{\varepsilon^j} \biggr) = 0,
\end{aligned}$$

because the second factor vanishes by the condition (3.7).

If the class K_α is infinite, then we again use the stationary equation (3.5) and a sufficient condition for the existence of the limit distribution in a closed class of states of a Markov process ([30], Chap. 3, § 6, [12]), namely, the existence of a nontrivial absolutely summable solution of the stationary second Kolmogorov system of equations. This completes the proof of Theorem 3.3.

Particular cases. The stationary distribution (3.8) for a system of interacting particles is connected with fundamental concepts of equilibrium statistical physics. Let the classes of communicating states be formed by the sets $K_E = \{\gamma \in \mathbb{N}^n : \gamma_1 + \dots + \gamma_n = E\}$, $E = 0, 1, 2, \dots$, and let there be a vector q satisfying the conditions (3.7). The limit distribution in the class K_E is determined by the generating function of a polynomial distribution,

$$\begin{aligned}
f_E(s) &= \biggl(\sum_{\gamma \in K_E} \frac{q^\gamma}{\gamma!} \biggr)^{-1} \biggl(\sum_{\gamma \in K_E} \frac{q^\gamma s^\gamma}{\gamma!} \biggr) \\
&= \biggl(\sum_{i=1}^{n} q_i \biggr)^{-E} \biggl(\sum_{i=1}^{n} q_i s_i \biggr)^{E} = (\widetilde{q}_1 s_1 + \dots + \widetilde{q}_n s_n)^E,
\end{aligned} \tag{3.9}$$

where $\widetilde{q}_i = q_i \bigl(\sum_{i=1}^{n} q_i \bigr)^{-1}$ for $i = 1, \dots, n$. The distribution (3.9) is the 'microcanonical' distribution [79], which holds for closed systems of interacting particles. If there were E particles of arbitrary types for $t = 0$, then as $t \to \infty$ the particles are distributed with respect to the types $T_1, \dots, T_n$ independently of one another with probability distribution $\{\widetilde{q}_1, \dots, \widetilde{q}_n\}$. The distribution (3.9) was obtained in [79] for bimolecular processes (in which case $|\varepsilon^i| = 2$, $i = 1, \dots, l$) under symmetry-type assumptions on the densities (3.6).

Let there be only one type T of particle and two interaction complexes $\varepsilon^1 = 0$ and $\varepsilon^2 = 1$, that is, let us consider the scheme of transmutations $0 \to T$ and $T \to 0$.

The entire set $\mathbb{N} = \{0, 1, \dots\}$ of states forms a closed class, and the stationary distribution is the Poisson distribution $\{q_{\alpha\gamma} = q^\gamma e^{-q}/\gamma!,\ \gamma \in \mathbb{N}\}$. This process is treated in [92] as an open system of interacting particles for which the Poisson distribution is interpreted as the 'canonical' distribution. Other special cases with two interaction complexes were discussed in [121].

Theorem 3.3 generalizes known results for branching processes with several types of particles, namely, assertions about the limit stationary distribution for a subcritical branching process with immigration ([106], Chap. 7, § 3) and the limit distribution in the final class of a branching process ([106], Chap. 4, § 7).

§ 3.3. Method of the exponential generating function

The exponential generating function was introduced in [39] to compute the probability of degeneration of a branching process with the interaction scheme $\varepsilon T \to kT$ (for $\varepsilon = 2, 3, \dots$); the probability of degeneration for the scheme $T \to kT$ was studied in [106], [33], and elsewhere. In [41] the method was applied to calculate the final probabilities of a process with the scheme $\varepsilon T_1 \to \gamma_1 T_1 + \gamma_2 T_2$ ($\varepsilon = 2, 3, \dots$), and limit theorems were obtained for the number of final particles of type T_2 when the initial number of particles of type T_1 is large; moreover, the scheme $T_1 \to \gamma_1 T_1 + \gamma_2 T_2$ studied in [106] (Chap. 5, §§ 4, 5) was generalized. Along with the results indicated below, the method of exponential generating function was developed in [51], [59], [58].

3.3.1. Population with two sexes. Let us consider a homogeneous Markov process $\xi(t) = (\xi_1(t), \xi_2(t))$, $t \in [0, \infty)$, on the state space $\mathbb{N}^2$ with the transition probabilities $P^{(\alpha_1,\alpha_2)}_{(\beta_1,\beta_2)}(t)$. Let the transition probabilities have the following form as $t \to 0+$ (for some $\lambda > 0$):

$$\begin{aligned} &P^{(\alpha_1,\alpha_2)}_{(\alpha_1,\alpha_2)}(t) = 1 - \alpha_1\alpha_2\lambda t + o(t), \quad \text{and} \\ &P^{(\alpha_1,\alpha_2)}_{(\beta_1,\beta_2)}(t) = p_{\beta_1-\alpha_1+1,\beta_2-\alpha_2+1}\alpha_1\alpha_2\lambda t + o(t) \end{aligned} \tag{3.10}$$

for $\alpha_1 \neq \beta_1$ or $\alpha_2 \neq \beta_2$. Here the probability distribution $\{p_{\gamma_1\gamma_2},\ p_{11} = 0\}$ is given. With the help of the generating functions ($|s_1| \leqslant 1$, $|s_2| \leqslant 1$)

$$F_{(\alpha_1,\alpha_2)}(t; s_1, s_2) = \sum_{\beta_1,\beta_2=0}^{\infty} P^{(\alpha_1,\alpha_2)}_{(\beta_1,\beta_2)}(t) s_1^{\beta_1} s_2^{\beta_2}, \quad h(s_1, s_2) = \sum_{\gamma_1,\gamma_2=0}^{\infty} p_{\gamma_1\gamma_2} s_1^{\gamma_1} s_2^{\gamma_2},$$

the second Kolmogorov system of differential equations for the process $\xi(t)$ can be represented as

$$\frac{\partial F_\alpha(t; s)}{\partial t} = \lambda(h(s_1, s_2) - s_1 s_2)\frac{\partial^2 F_\alpha(t; s)}{\partial s_1 \partial s_2}, \quad F_\alpha(0; s) = s^\alpha.$$

Corresponding to a process $\xi(t)$ of class B_2 is the interaction scheme $T_1 + T_2 \to \gamma_1 T_1 + \gamma_2 T_2$. Let the process be in the initial state (α_1, α_2). Following § 1.3.1, we can assume that in a random time $\tau_{(\alpha_1,\alpha_2)}$ with $\mathsf{P}\{\tau_{(\alpha_1,\alpha_2)} \leqslant t\} = 1 - e^{-\alpha_1\alpha_2\lambda t}$ there is an interaction of a particle of type T_1 with a particle of type T_2. These two

particles transform into a group of γ_1 particles of type T_1 and γ_2 particles of type T_2 with probability distribution $\{p_{\gamma_1\gamma_2}\}$; the process passes to the state corresponding to the vector $(\alpha_1+\gamma_1-1, \alpha_2+\gamma_2-1)$. The further evolution of the stochastic process is similar. The branching process $\xi(t)$ is a model of a population with male individuals and female individuals ([118], [58]). The main assumptions of the model are as follows: any pair of individuals T_1+T_2 in the population generates descendants independently of the others; the frequency of acts of generation of new individuals is proportional both to the number of individuals of type T_1 and to the number of individuals of type T_2.

The states $(\gamma_1, 0)$, $(0, \gamma_2)$, $\gamma_1, \gamma_2 \in \mathbb{N}$, are absorbing for the process $\xi(t)$ (particles of only one type remain, and no interaction is possible). From symmetry arguments it suffices to consider the final probabilities of the form

$$q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)} = \lim_{t\to\infty} P_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)}(t), \qquad \gamma_2 \in \mathbb{N}.$$

The exponential generating function

$$g_{(0,\gamma_2)}(z_1, z_2) = \sum_{\alpha_1,\alpha_2=0}^{\infty} \frac{z_1^{\alpha_1} z_2^{\alpha_2}}{\alpha_1!\,\alpha_2!} q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)} \tag{3.11}$$

satisfies the stationary first Kolmogorov equation (the equation (3.2)),

$$\left[h\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right) - \frac{\partial^2}{\partial z_1 \partial z_2}\right] g_{(0,\gamma_2)}(z_1, z_2) = 0. \tag{3.12}$$

It follows from obvious equalities for the final probabilities (of the form $q_{(0,\gamma_2)}^{(0,\gamma_2)} = 1$, $q_{(0,\gamma_2)}^{(0,\alpha_2)} = 0$ for $\alpha_2 \neq \gamma_2$, and $q_{(0,\gamma_2)}^{(\alpha_1,0)} = 0$ for $\alpha_1 = 0, 1, \dots$) that the following boundary conditions hold:

$$g_{(0,\gamma_2)}(0, z_2) = \frac{z_2^{\gamma_2}}{\gamma_2!}, \qquad g_{(0,\gamma_2)}(z_1, 0) = 0. \tag{3.13}$$

The following is the characteristic equation for the partial differential equation (3.12) with constant coefficients:

$$h(s_1, s_2) - s_1 s_2 = 0 \tag{3.14}$$

(cf. [106], Chap. 2, § 1, (9) and [33], Chap. 5, § 4, (2) for the scheme $T \to kT$).

The problem of finding the probability $q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)}$ coincides with the problem of the stopping probability at the boundary for a homogeneous random walk in the plane quadrant $\mathbb{N}^2$ [85] (see also [90]), in which the main role is played by the multivalued function (of a complex variable) defined by the equation (3.14) ([84], [25]). The analytic apparatus developed in [84] clarified the role of elliptic functions in the solution of problems on multidimensional walks with boundaries. In [31] an integral representation for $q_{(0,\gamma_2)}^{(1,\alpha_2)}$ containing the functions $\operatorname{cn} u$ and $\operatorname{sn} u$ was found in the symmetric case $h(s_1, s_2) = p_{01}s_1^2 s_2 + p_{10}s_1 s_2^2 + p_{10}s_1 + p_{01}s_2$ by using the methods

of [84]. In [85] one can find references concerning the boundary-value problem of the form (3.12), (3.13).

Integral representation for the final probabilities in the case $h(s_1, s_2) = p_{20}s_1^2 + p_{02}s_2^2 + p_{00}$. The equation (3.12) becomes

$$p_{20}\,\frac{\partial^2 g_{(0,\gamma_2)}}{\partial z_1^2} + p_{02}\,\frac{\partial^2 g_{(0,\gamma_2)}}{\partial z_2^2} + p_{00}\,g_{(0,\gamma_2)} - \frac{\partial^2 g_{(0,\gamma_2)}}{\partial z_1 \partial z_2} = 0 \tag{3.15}$$

with the boundary conditions (3.13). The reduction of the equation (3.15) of hyperbolic type to the canonical form gives the telegraph equation. For (3.15) we obtain a Darboux–Picard problem, namely, the conditions (3.13) are given on two monotone curves going out from the point $(0,0)$ and staying in a characteristic angle with vertex at the same point $(0,0)$. In [61], on the basis of the Riemann formula for the telegraph equation ([11], [113]), a solution of the boundary-value problem (3.15), (3.13) was constructed in the form of a series in Bessel functions of integral order. After some manipulations, the solution is summed to an integral representation containing an elliptic function. We present the main result of [51], [61].

Let ω and ω' be positive numbers. We set $h = e^{-\pi\omega'/\omega}$ and $z = e^{\pi i u/(2\omega)}$. Let us define the doubly periodic function (for $\gamma = 1, 2, \dots$)

$$f_\gamma(u) = \left(\frac{\pi i}{\omega}\right)^\gamma \left[\frac{1}{(z - z^{-1})^\gamma} + \sum_{r=1}^{\infty}\left(\frac{h^r z^{-1}}{1 - h^{2r}z^{-2}}\right)^\gamma + \sum_{r=1}^{\infty}\left(\frac{-h^r z}{1 - h^{2r}z^2}\right)^\gamma\right]. \tag{3.16}$$

The periods are 4ω and $2i\omega'$ for odd γ and 2ω and $2i\omega'$ for even γ. Up to a constant summand, the function $f_2(u)$ coincides with the Weierstrass elliptic function ([35], Chap. 2, § 12), and the convergence of the series (3.16) was studied in [35].

Theorem 3.4. *Let a Markov process on the state space* $\mathbb{N}^2$ *be given by the relations* (3.10) *and let* $h(s_1, s_2) = p_{20}s_1^2 + p_{02}s_2^2 + p_{00}$. *Let*

$$C = \frac{1 - \sqrt{1 - 4p_{20}p_{02}}}{1 + \sqrt{1 - 4p_{20}p_{02}}}, \qquad C_0 = \frac{p_{20}p_{00}}{1 - 4p_{20}p_{02}}.$$

Let $f_{\gamma_2}(u)$ *be the elliptic function* (3.16) *with the half-periods* $\omega = \pi\sqrt{C_0}$ *and* $\omega' = -\sqrt{C_0}\ln C$, *and let* $T = \{u = x + iy,\ 0 \leqslant x \leqslant 4\omega,\ y = -\omega'\}$ *be a segment oriented with respect to increasing* x. *The exponential generating function of the final probabilities is*

$$g_{(0,\gamma_2)}(z_1, z_2) = \frac{1}{2\pi i}\int_T e^{z_1 s_1(z) + z_2 s_2(z)}\,\frac{(-1)^{\gamma_2}}{-\gamma_2}\,df_{\gamma_2}(u), \quad \gamma_2 = 1, 2, \dots, \tag{3.17}$$

where $z = e^{\pi i u/(2\omega)}$, *and the functions*

$$s_1(z) = \frac{1 - \sqrt{1 - 4p_{20}p_{02}}}{2p_{20}}\, i\sqrt{C_0}\left(z - \frac{1}{Cz}\right), \qquad s_2(z) = i\sqrt{C_0}\left(z - \frac{1}{z}\right),$$

define a uniformization of the Riemann surface (3.14).

We denote by $\eta^{(\alpha_1,\alpha_2)}$ the number of final particles of type T_2 remaining after degeneration of the process, that is, when there are no particles of type T_1. The random variable $\eta^{(\alpha_1,\alpha_2)}$ has the distribution $\{q^{(\alpha_1,\alpha_2)}_{(0,\gamma_2)}, \gamma_2 \in \mathbb{N}; \sum_{\gamma_2=0}^{\infty} q^{(\alpha_1,\alpha_2)}_{(0,\gamma_2)} \leqslant 1\}$; the expressions for $q^{(\alpha_1,\alpha_2)}_{(0,\gamma_2)}$ follow from the definition (3.11) of the function $g_{(0,\gamma_2)}(z_1, z_2)$ and from (3.17). The explicit formulae for the final probabilities provide proofs of limit theorems of interest in applications (see [106], [84]) for the number of final particles as $\alpha_1 \to \infty$ and $\alpha_2 \to \infty$.

In [61] solutions of the equation (3.12) were obtained under other assumptions about the generating function $h(s_1, s_2)$ of the distribution of the number of descendants of an interacting pair of particles.

3.3.2. Epidemic processes. Let $\xi(t) = (\xi_1(t), \xi_2(t))$, $t \in [0, \infty)$, be a Markov process with transition probabilities $P^{(\alpha_1,\alpha_2)}_{(\beta_1,\beta_2)}(t)$ on the set $\mathbb{N}^2$, and let the transition probabilities be of the following form as $t \to 0+$ (for $\mu > 0$):

$$\begin{aligned} &P^{(\alpha_1,\alpha_2)}_{(\alpha_1,\alpha_2)}(t) = 1 - (\alpha_1\alpha_2 + \mu\alpha_1)t + o(t), \quad \text{and} \\ &P^{(\alpha_1,\alpha_2)}_{(\beta_1,\beta_2)}(t) = (p^2_{\beta_1-\alpha_1+1,\beta_2-\alpha_2+1}\alpha_1\alpha_2 + p^1_{\beta_1-\alpha_1+1,\beta_2-\alpha_2}\mu\alpha_1)t + o(t) \end{aligned} \tag{3.18}$$

if $\alpha_1 \neq \beta_1$ or $\alpha_2 \neq \beta_2$. Let the probability distributions $\{p^2_{\gamma_1\gamma_2}, p^2_{11} = 0\}$ and $\{p^1_{\gamma_1\gamma_2}, p^1_{10} = 0\}$ be given. The second Kolmogorov equation for the generating function $F_{(\alpha_1,\alpha_2)}(t; s_1, s_2)$ in the case of the process $\xi(t)$ becomes

$$\frac{\partial F_\alpha(t;s)}{\partial t} = \big(h_2(s_1,s_2) - s_1 s_2\big)\frac{\partial^2 F_\alpha(t;s)}{\partial s_1 \partial s_2} + \mu\big(h_1(s_1,s_2) - s_1\big)\frac{\partial F_\alpha(t;s)}{\partial s_1}, \; F_\alpha(0;s) = s^\alpha;$$

the interaction scheme is $T_1 \to \gamma_1^1 T_1 + \gamma_2^1 T_2$, $T_1 + T_2 \to \gamma_1^2 T_1 + \gamma_2^2 T_2$.

An event $\{\xi(t) = (\beta_1, \beta_2)\}$ means the existence of β_1 particles of type T_1 and β_2 particles of type T_2 at the time t. The following description is customary in probabilistic models of the spreading of an epidemic ([6], [4], [24], [5], [115]). The particles of type T_1 are interpreted as sick individuals and the particles of type T_2 as healthy individuals susceptible to the infectious disease. In a random time $\tau^2_{(\beta_1,\beta_2)}$ with $\mathsf{P}\{\tau^2_{(\beta_1,\beta_2)} \leqslant t\} = 1 - e^{-\beta_1\beta_2 t}$ there is contact of a particle of type T_1 with a particle of type T_2. This pair of particles changes into a new group of γ_1 particles of type T_1 and γ_2 particles of type T_2 with the distribution $\{p^2_{\gamma_1\gamma_2}\}$; the process goes to the state corresponding to the vector $(\beta_1 + \gamma_1 - 1, \beta_2 + \gamma_2 - 1)$. Moreover, in a random time $\tau^1_{(\beta_1,\beta_2)}$ with $\mathsf{P}\{\tau^1_{(\beta_1,\beta_2)} \leqslant t\} = 1 - e^{-\mu\beta_1 t}$ a particle of type T_1 changes into a group of particles with distribution $\{p^1_{\gamma_1\gamma_2}\}$; the process goes to the state corresponding to the vector $(\beta_1 + \gamma_1 - 1, \beta_2 + \gamma_2)$. The random variables $\tau^1_{(\beta_1,\beta_2)}$ and $\tau^2_{(\beta_1,\beta_2)}$ are independent; the process stays in the state (β_1, β_2) during the random time $\tau_{(\beta_1,\beta_2)} = \min(\tau^1_{(\beta_1,\beta_2)}, \tau^2_{(\beta_1,\beta_2)})$.

In [22] the branching process $\xi(t)$ was introduced as a model of a chain reaction generating neutrons (particles of type T_1) with nuclei of heavy elements (particles of type T_2) taken into account, under the assumptions that $h_2(s_1, s_2) = h_2(s_1)$ and $h_1(s_1, s_2) = 1$.

The states $(0,\gamma_2)$, $\gamma_2 \in \mathbb{N}$, are absorbing for the process $\xi(t)$ (the remaining individuals are healthy). For the final probabilities $q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)} = \lim_{t\to\infty} P_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)}(t)$, $\gamma_2 \in \mathbb{N}$, we introduce (for $|s| \leqslant 1$) the generating functions

$$f_{(\alpha_1,\alpha_2)}(s) = \sum_{\gamma_2=0}^{\infty} q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)} s^{\gamma_2}, \quad \Phi(z_1,z_2;s) = \sum_{\alpha_1,\alpha_2=0}^{\infty} \frac{z_1^{\alpha_1} z_2^{\alpha_2}}{\alpha_1!\,\alpha_2!} f_{(\alpha_1,\alpha_2)}(s). \tag{3.19}$$

As in Theorem 3.1, we can find the equation for the double generating function by using the equation (1.18),

$$\left[z_2\left(h_1\left(\frac{\partial}{\partial z_1},\frac{\partial}{\partial z_2}\right) - \frac{\partial^2}{\partial z_1 \partial z_2}\right) + \mu\left(h_2\left(\frac{\partial}{\partial z_1},\frac{\partial}{\partial z_2}\right) - \frac{\partial}{\partial z_1}\right)\right]\Phi(z_1,z_2;s) = 0. \tag{3.20}$$

The equalities for the final probabilities ($q_{(0,\gamma_2)}^{(0,\gamma_2)} = 1$ and $q_{(0,\alpha_2)}^{(0,\gamma_2)} = 0$ for $\alpha_2 \neq \gamma_2$) imply the boundary condition $\Phi(0,z_2;s) = e^{z_2 s}$.

Integral representation for the final probabilities in the case $h_2(s_1,s_2) = s_1$ and $h_1(s_1,s_2) = 1$. For the Weiss epidemic process ([126], [114]) the equation (3.20) becomes

$$z_2\Phi_{z_1 z_2} + (\mu - z_2)\Phi_{z_1} - \mu\Phi = 0. \tag{3.21}$$

If the initial state of the process under consideration is of the form $(\alpha_1,0)$, then a jump to the state $(\alpha_1-1,0)$ occurs; hence, the final probabilities are $q_{(0,0)}^{(\alpha_1,0)} = 1$ for $\alpha_1 = 0,1,\dots$ and $q_{(0,\gamma_2)}^{(\alpha_1,0)} = 0$ for $\alpha_1 = 0,1,\dots$ and $\gamma_2 = 1,2,\dots$. Therefore, $\Phi(z_1,0;s) = e^{z_1}$. Thus, we obtain a Goursat problem for the hyperbolic equation (3.21), namely, the boundary conditions are given on the characteristics $z_1 = 0$ and $z_2 = 0$ in the form $\Phi(0,z_2;s) = e^{z_2 s}$ and $\Phi(z_1,0;s) = e^{z_1}$. Partial differential equations of hyperbolic type whose coefficients are linear functions of the independent variables were studied in [19] and [103], § 41. In [43] the author obtained the Riemann function for the equation (3.21) and the explicit solution

$$\Phi(z_1,z_2;s) = \int_0^\infty \left(1 + \frac{1}{\mu}(s-1)e^{-x/\mu} z_2\right) e^{-x+(1-e^{-x/\mu}+se^{-x/\mu})z_2} J_0\big(2\sqrt{-z_1 x}\,\big)\,dx. \tag{3.22}$$

The definition in (3.19) of the function $\Phi(z_1,z_2;s)$ and the formula (3.22), together with the power series expansions

$$e^z = \sum_{i=0}^{\infty} \frac{z^i}{i!}, \qquad J_0(z) = \sum_{j=0}^{\infty} \frac{(-1)^j (z/2)^{2j}}{j!\,j!},$$

imply the integral representation

$$f_{(\alpha_1,\alpha_2)}(s) = \frac{1}{\alpha_1!}\int_0^\infty x^{\alpha_1}\Big[(1-e^{-x/\mu}+se^{-x/\mu})^{\alpha_2} + \frac{1}{\mu}\alpha_2(s-1)e^{-x/\mu}(1-e^{-x/\mu}+se^{-x/\mu})^{\alpha_2-1}\Big]e^{-x}\,dx.$$

Integrating the second summand by parts, we obtain the following assertion.

Theorem 3.5 [43]. *Let a Markov process on the state space* $\mathbb{N}^2$ *be given by the relations* (3.18) *and let* $h_2(s_1,s_2)=s_1$ *and* $h_1(s_1,s_2)=1$. *The generating function of the final probabilities has the form* (*for* $\alpha_1\neq 0$)

$$f_{(\alpha_1,\alpha_2)}(s)=\frac{1}{(\alpha_1-1)!}\int_0^\infty x^{\alpha_1-1}(1-e^{-x/\mu}+s\,e^{-x/\mu})^{\alpha_2}\,e^{-x}\,dx. \tag{3.23}$$

Let $\eta^{(\alpha_1,\alpha_2)}$ be the number of final particles of type T_2 that remain after stopping the process. The generating function (3.23) defines the distribution of the random variable $\eta^{(\alpha_1,\alpha_2)}$ on the states $\{(0,\gamma_2),\ \gamma_2=0,\dots,\alpha_2\}$. It follows from (3.23) by virtue of the formulae (1.4) and (1.5) that for the mean and variance we have

$$\mathsf{E}\eta^{(\alpha_1,\alpha_2)}=\alpha_2\biggl(\frac{\mu}{1+\mu}\biggr)^{\alpha_1},\qquad \mathsf{D}\eta^{(\alpha_1,\alpha_2)}\sim\alpha_2^2\biggl(\biggl(\frac{\mu}{2+\mu}\biggr)^{\alpha_1}-\biggl(\frac{\mu}{1+\mu}\biggr)^{2\alpha_1}\biggr)$$

as $\alpha_2\to\infty$. Applying the method of characteristic functions to the expression (3.23) in the standard way (cf. [106], Chap. 5, § 5, [41]), we obtain a limit theorem of 'threshold' type ([6], [114]).

Theorem 3.6. *Let the assumptions of Theorem* 3.5 *hold and let* $x\in[0,1]$. *Then*

$$\lim_{\alpha_2\to\infty}\mathsf{P}\biggl\{\frac{\eta^{(\alpha_1,\alpha_2)}}{\alpha_2}\leqslant x\biggr\}=\frac{1}{(\alpha_1-1)!}\int_{-\mu\ln x}^\infty y^{\alpha_1-1}e^{-y}\,dy.$$

A series of limit theorems for $\eta^{(\alpha_1,\alpha_2)}$ was established in [114].

In [43] a solution of (3.20) was obtained for $h_2(s_1,s_2)=p_{10}^2s_1+p_{01}^2s_2+p_{00}^2$ and $h_1(s_1,s_2)=1$.

Deterministic approximation. The relationships between the probabilistic and deterministic descriptions for diverse (Markov) epidemic processes were studied in [6], [10], [4], and elsewhere. Corresponding to a Weiss epidemic process is a deterministic model in the form of the system

$$\dot x_1=-\mu x_1,\qquad \dot x_2=-x_1x_2$$

of differential equations [126], where $x_1(t)$ is the number of sick individuals and $x_2(t)$ is the number of individuals susceptible to the infectious disease.

3.3.3. Open problems. Below we formulate problems on stationary and final probabilities for Markov processes of class B_2 with schemes that are closest to those of processes of class B_1, which were studied in [106]. Formulation of the problems for general interaction schemes is possible if the processes of class B_2 can be classified.

1. A classification of the types of particles for branching processes of class B_1 was given in [106] (Chap. 4, § 6). A problem is to determine a similar classification of the types of particles for branching processes of class B_2.

2. For a process with immigration of class B_2 with the scheme $0\to k_0T$, $T\to k_1T$, $2T\to k_2T$ ([106], Chap. 7, [121], § 9.1, [74]) the second Kolmogorov equation is

$$\frac{\partial F_i(t;s)}{\partial t}=\lambda(h_2(s)-s^2)\frac{\partial^2F_i(t;s)}{\partial s^2}+\mu(h_1(s)-s)\frac{\partial F_i(t;s)}{\partial s}+\rho(h_0(s)-1)F_i(t;s)$$

(where $\lambda \geqslant 0$, $\mu \geqslant 0$, $\rho > 0$), with the initial condition $F_i(0;s) = s^i$. Let the mean number of particles appearing under pairwise interaction be less than two, that is, the criticality parameter a_2 satisfies $a_2 = \lambda(h_2'(1) - 2) < 0$. A problem is to find explicit expressions for the stationary distribution of the Markov process under particular assumptions about the generating functions $h_0(s)$, $h_1(s)$, and $h_2(s)$.

3. The first Kolmogorov equation for a branching process with two interaction complexes $T \to k_1 T$ and $2T \to k_2 T$ is [59]

$$\frac{\partial G_j(t;z)}{\partial t} = \left[\lambda z^2\left(h_2\left(\frac{\partial}{\partial z}\right) - \frac{\partial^2}{\partial z^2}\right) + \mu z\left(h_1\left(\frac{\partial}{\partial z}\right) - \frac{\partial}{\partial z}\right)\right] G_j(t;z), \quad G_j(0;z) = \frac{z^j}{j!}\,.$$

A problem is to get an integral representation for the degeneration probability q_{i0}.

4. The first equation for a process of class B_2 with the interaction scheme $T_1 \to \gamma_1^1 T_1 + \gamma_2^1 T_2$ and $2T_2 \to \gamma_1^2 T_1 + \gamma_2^2 T_2$ is

$$\frac{\partial G_\beta(t;z)}{\partial t} = \left[\lambda z_2^2\left(h_2\left(\frac{\partial}{\partial z}\right) - \frac{\partial^2}{\partial z_2^2}\right) + \mu z_1\left(h_1\left(\frac{\partial}{\partial z}\right) - \frac{\partial}{\partial z_1}\right)\right] G_\beta(t;z),$$
$$G_\beta(0;z) = \frac{z^\beta}{\beta!}\,.$$

A problem is to find the degeneration probabilities $q_{(0,0)}^{(\alpha_1,\alpha_2)}$ and $q_{(0,1)}^{(\alpha_1,\alpha_2)}$.

5. Let us consider the means $A_1(t) = \mathsf{E}\xi_1(t)$ and $A_2(t) = \mathsf{E}\xi_2(t)$ for the branching process $\xi(t) = (\xi_1(t), \xi_2(t))$ in § 3.3.1. If $h(s_1, s_2) = s_2 h_1(s_1)$, then we can readily see for the initial state (α_1, α_2) of the process that $A_1(t) = \alpha_1 e^{\alpha_2 a_1 t}$ and $A_2(t) = \alpha_2$, where $a_1 = \lambda((\partial h_1/\partial s_1)|_{s=1} - 1)$ is the criticality parameter. A problem is to study the asymptotic behaviour of the means $A_1(t)$ and $A_2(t)$ as $t \to \infty$ for other special assumptions about the generating function $h(s_1, s_2)$ and to find an integral representation for the final probabilities $q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)}$ for an arbitrary function $h(s_1, s_2)$.

6. For the Bartlett–McKendrick epidemic process we have $h_2(s_1, s_2) = s_1^2$ and $h_1(s_1, s_2) = 1$ ([6], [24]), and the stationary equation (3.20) becomes

$$z_2\Phi_{z_1 z_2} - z_2\Phi_{z_1 z_1} + \mu\Phi_{z_1} - \mu\Phi = 0$$

with the boundary conditions $\Phi(0, z_2; s) = e^{z_2 s}$ and $\Phi(z_1, 0; s) = e^{z_1}$. A problem is to find the Riemann function and an integral representation for the generating function $\Phi(z_1, z_2; s)$. The cumbersome expression obtained in [28], [111] for the final probabilities $q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)}$ is of little use for the asymptotic investigation.

Another problem is to solve the equation (3.20) for $h_2(s_1, s_2) = p_{20}^2 s_1^2 + p_{02}^2 s_2^2 + p_{10}^2 s_1 + p_{01}^2 s_2 + p_{00}^2$ and $h_1(s_1, s_2) = p_2^1 s_1^2 + p_0^1$ with the boundary conditions $\Phi(0, z_2; s) = e^{z_2 s}$ and $\Phi(z_1, 0; s) = e^{z_1 q}$, where q is the root nearest to zero of the equation $p_2^1 s_1^2 + p_0^1 - s_1 = 0$.

7. For the predator-prey Markov process in § 2.3 a problem is to find an integral representation for the degeneration probability $q_{(0,0)}^{(\alpha_1,\alpha_2)}$ by considering the stationary first Kolmogorov equation.

CHAPTER IV

THIRD KOLMOGOROV EQUATION

The transition probabilities of Markov processes with countable state space satisfy the first and second systems of differential equations (1.2) and (1.3), which are *linear*. The Markov branching process introduced in [71] was described as a process of evolution of particles; under the assumption that the individual evolving particles are independent of one another, a *non-linear* first-order differential equation was obtained in [71] for the generating functions of the transition probabilities of such a process. The authors of the paper stressed that "... the remark shows that our 'branching stochastic processes' are in fact only a special case of *Markov processes with countable state space*. However, we shall obtain an analytic apparatus for this special case which is much more efficient than the apparatus that can be developed for the general case of Markov processes with countable state space" (italics due to the authors of [71]). Thus, the paper [71] poses the following questions. Are there other special cases of Markov processes with countable state space whose transition probabilities satisfy a non-linear equation? If there are examples of Markov processes of this kind, then how can one distinguish possible special classes of Markov processes whose transition probabilities satisfy non-linear equations of diverse types in the set of all Markov processes?

Methods of investigation of branching stochastic processes have been intensively developed [106]. Some statements of the theory of Markov branching processes are briefly presented in § 4.1, where one of the possible ways of deriving a non-linear equation for the generating function of the transition probabilities of the process is given. The non-linear equation is a consequence of the branching property.

In § 4.2 we derive a third (non-linear) equation for the processes of class B_2 with pair interactions of the form $2T \to kT$, $k = 0, 1$, and $2T \to 3T$; the exact solutions of the first and second Kolmogorov equations are known for such processes of quadratic type. A generalized branching property of the transition probabilities is discovered by constructing closed solutions of partial differential equations of parabolic type; we use separation of variables and summation formulae from the theory of special functions. The non-linear differential equations thus obtained are first-order partial differential equations. The ways described for constructing solutions of the linear Kolmogorov equations can be applied to one-dimensional and multidimensional birth-and-death Markov processes of the types indicated in § 2.1.

§ 4.1. Non-linear equation of the theory of branching processes

On the state space $\mathbb{N} = \{0, 1, 2, \dots\}$ we consider a time-homogeneous Markov process ξ_t for $t \in [0, \infty)$ with transition probabilities $P_{ij}(t)$, and we define the infinitesimal characteristics $a_{ij} = (dP_{ij}(t)/dt)|_{t=0+}$, $i, j \in \mathbb{N}$, by the equalities (for $\lambda > 0$)

$$a_{ij} = i\lambda p_{j-i+1}, \ \ j \geqslant i-1, \ \ j \neq i, \quad a_{ii} = -i\lambda, \quad a_{ij} = 0, \ \ j < i-1,$$

for a given probability distribution $\{p_k,\ k \in \mathbb{N};\ p_1 = 0\}$. With the help of the generating functions

$$F_i(t; s) = \sum_{j=0}^{\infty} P_{ij}(t) s^j, \quad i \in \mathbb{N}, \qquad h(s) = \sum_{k=0}^{\infty} p_k s^k, \tag{4.1}$$

where $|s| < 1$, the second system of equations can be represented as

$$\frac{\partial F_i(t;s)}{\partial t} = \lambda(h(s) - s)\frac{\partial F_i(t;s)}{\partial s}\,, \qquad F_i(0;s) = s^i. \tag{4.2}$$

The double generating function of the transition probabilities

$$\mathcal{F}(t;z;s) = \sum_{i=0}^{\infty} \frac{z^i}{i!} F_i(t;s) = \sum_{i,j=0}^{\infty} \frac{z^i}{i!} P_{ij}(t) s^j = \sum_{j=0}^{\infty} G_j(t;z) s^j \tag{4.3}$$

satisfies the Kolmogorov equations (1.18) and (1.19):

$$\frac{\partial \mathcal{F}}{\partial t} = \lambda z\left(h\left(\frac{\partial}{\partial z}\right) - \frac{\partial}{\partial z}\right)\mathcal{F}, \tag{4.4}$$

$$\frac{\partial \mathcal{F}}{\partial t} = \lambda(h(s) - s)\frac{\partial \mathcal{F}}{\partial s}\,, \quad \mathcal{F}(0;z;s) = e^{zs}. \tag{4.5}$$

Solving the first-order linear partial differential equation (4.2) by standard methods [63], we obtain the *branching property* of the stochastic process under consideration:

$$F_i(t;s) = F_1^i(t;s), \qquad i \in \mathbb{N}, \tag{4.6}$$

that is, the transition probabilities satisfy the condition (see [106], Chap. 1, § 1, (5))

$$P_{ij}(t) = \sum_{j_1+j_2+\cdots+j_i=j} P_{1j_1}(t) P_{1j_2}(t) \cdots P_{1j_i}(t). \tag{4.7}$$

Indeed, let the function $F_1(t;s)$ be a solution of the equation (4.2) with the initial condition s; then by substitution in (4.2) we can readily see that the function $F_1^i(t;s)$ satisfies (4.2) with the initial condition s^i. The branching property (4.6), together with the uniqueness condition $a = \lambda(h'(1) - 1) < \infty$ for a solution of (4.2), was studied in [33], (Chap. 5, § 4).

It follows from (4.6) that the double generating function satisfies the equality

$$\mathcal{F}(t;z;s) = \sum_{i=0}^{\infty} \frac{z^i}{i!} F_1^i(t;s) = e^{zF_1(t;s)}. \tag{4.8}$$

Substituting the expression (4.8) in the first equation of (4.4), we obtain the following ordinary differential equation for the generating function $F_1(t;s)$ (see [106], Chap. 1, § 4, Theorem 5 and [33], Chap. 5, § 9, (9.1)):

$$\frac{\partial F_1(t;s)}{\partial t} = \lambda\big(h(F_1(t;s)) - F_1(t;s)\big), \qquad F_1(0;s) = s. \tag{4.9}$$

Thus, for Markov process ξ_t of class B_1 we have three equations: the first and second Kolmogorov equations and the non-linear equation (4.9).

Corresponding to the process ξ_t is the transmutation scheme $T \to kT$; a state j of the process is interpreted as the existence of j particles of type T. A particle can

have descendants, that is, an aggregate of k particles of type T with the probability distribution $\{p_k\}$; each of i initial particles has randomly many descendants $\xi_t^{(l)}$ ($l = 1, \dots, i$) at a time t, and the following relation holds ([106], Chap. 1, § 2):

$$\xi_t = \xi_t^{(1)} + \dots + \xi_t^{(i)}. \tag{4.10}$$

The branching property of the transition probabilities (4.7) means that the random variables $\xi_t^{(l)}$, $l = 1, \dots, i$, are independent and identically distributed. The general theory of branching processes includes the construction of both the space of elementary events that consists of a description of the evolution of the individual particles that existed during the time interval $[0, t]$, which is a set of trees, and the corresponding probability measure on the trees ([106], Chap. 12). In Fig. 5 (§ 5.3) we show an example of a realization of a branching process on this state space. The relationship of the characteristics of branching processes with those of random trees and forests was studied in [67] (see also the survey in [68], § 1.9).

It follows from the first and second equations (4.4) and (4.5) that

$$(h(s) - s)\frac{\partial \mathcal{F}}{\partial s} - z\left(h\left(\frac{\partial}{\partial z}\right) - \frac{\partial}{\partial z}\right)\mathcal{F} = 0.$$

Substituting the expression (4.8) into the last equation, we obtain the following equation for the generating function $F_1(t; s)$:

$$(h(s) - s)\frac{\partial F_1}{\partial s} - (h(F_1) - F_1) = 0. \tag{4.11}$$

Let us note special cases of the branching property (4.6). It follows from the explicit solutions of the Kolmogorov equations (see 2.1.1) that for a simple death process of linear type for which $h(s) = 1$ ($a_{i,i-1} = i\lambda$, $a_{ii} = -i\lambda$) we have

$$F_i(t; s) = (1 - e^{-\lambda t} + s\, e^{-\lambda t})^i, \qquad i \in \mathbb{N}. \tag{4.12}$$

For a pure birth process of linear type with $h(s) = s^2$ ($a_{ii} = -i\lambda$, $a_{i,i+1} = i\lambda$) we have

$$F_i(t; s) = \left(\frac{s\, e^{-\lambda t}}{1 - s\,(1 - e^{-\lambda t})}\right)^i, \qquad i \in \mathbb{N}. \tag{4.13}$$

For a birth-and-death process of linear type with $h(s) = p_0 + p_2 s^2$ ($a_{i,i-1} = i\lambda p_0$, $a_{ii} = -i\lambda$, and $a_{i,i+1} = i\lambda p_2$) the expression for $F_i(t; s)$ is given by the formula (2.11).

§ 4.2. Branching process with pair interactions

Let us consider a Markov process ξ_t of class B_2 on the state space $\mathbb{N}$ for $t \in [0, \infty)$. We define the infinitesimal characteristics of the process as follows (for $\lambda > 0$):

$$a_{ij} = i(i-1)\lambda p_{j-i+2}, \;\; j \geqslant i-2, \;\; j \neq i, \quad a_{ii} = -i(i-1)\lambda, \quad a_{ij} = 0, \;\; j < i-2, \tag{4.14}$$

where $\{p_k,\ k \in \mathbb{N};\ p_2 = 0\}$ is some probability distribution. The generating function (4.1) of the transition probabilities $P_{ij}(t)$ of the process ξ_t satisfies the second-order partial differential equation of parabolic type

$$\frac{\partial F_i(t;s)}{\partial t} = \lambda(h(s) - s^2)\frac{\partial^2 F_i(t;s)}{\partial s^2}, \qquad F_i(0;s) = s^i. \tag{4.15}$$

The double generating function (4.3) satisfies the Kolmogorov equations

$$\frac{\partial \mathcal{F}}{\partial t} = \lambda z^2\left(h\left(\frac{\partial}{\partial z}\right) - \frac{\partial^2}{\partial z^2}\right)\mathcal{F}, \qquad \frac{\partial \mathcal{F}}{\partial t} = \lambda(h(s) - s^2)\frac{\partial^2 \mathcal{F}}{\partial s^2}, \qquad \mathcal{F}(0;z;s) = e^{zs}.$$

Hence,

$$(h(s) - s^2)\frac{\partial^2 \mathcal{F}}{\partial s^2} - z^2\left(h\left(\frac{\partial}{\partial z}\right) - \frac{\partial^2}{\partial z^2}\right)\mathcal{F} = 0. \tag{4.16}$$

The investigation of the equation (4.15) was begun in [104]. The Cauchy problem (4.15) admits a solution having a probabilistic meaning, as follows from the existence of a solution of the second system of differential equations for the transition probabilities ([26], [30]). In [104] it was shown that if the criticality parameter a satisfies the condition $a = \lambda(h'(1) - 2) \leqslant 0$, then the solution of problem (4.15) is unique, and the following *regularity condition* for the branching process is satisfied ([106], Chap. 1, § 5): $\lim_{s\uparrow 1} F_i(t;s) \equiv 1$.

In [3] the asymptotic behaviour as $t \to \infty$ of the probability $Q_i(t) = \mathsf{P}\{\xi_t > 0 \mid \xi_0 = i\} = 1 - F_i(t;0)$ of continuation was studied for a critical branching process ($a = 0$). It was shown that $Q_2(t) = o(e^{-\delta t})$ for some $\delta > 0$; methods of operational calculus were applied in [3] to (4.15). In [54], [55] an explicit solution of (4.15) was found for $h(s) = s^4/2 + 1/2$, and the exact asymptotics for $Q_i(t)$ was given in this case.

Below we present results for which a preliminary exposition was given in [47], [50]. The branching property (4.6) plays the main role in the derivation of the nonlinear equation of the theory of branching processes. The property (4.6) fails to hold for the transition probabilities of a branching process ξ_t with the scheme $2T \to kT$; however, an analogue of (4.6) was discovered and a generalization of the equation (4.11) can be given.

Deterministic approximation. Branching processes with the schemes $2T \to 0$ ($h(s) = 1$) and $2T \to T$ ($h(s) = s$) for large initial numbers of particles were studied in [88], [89] in comparison with the law of active mass. If $x(t)$ is the amount of the reagent at time t for a bimolecular reaction with the scheme $2T \to kT$, $k = 0, 1$ (in the probabilistic model $h(s) = p_0 + p_1 s$), then in formal kinetics the law assumed is [23]

$$\dot{x} = ax^2,$$

where a is the reaction rate constant ($a = \lambda(h'(1) - 2) < 0$).

4.2.1. Solution of the first and second equations for a death process of quadratic type. The construction of a closed solution of the linear equation

$$\frac{\partial F_i(t;s)}{\partial t} = \lambda(p_0 + p_1 s - s^2)\frac{\partial^2 F_i(t;s)}{\partial s^2}, \qquad F_i(0;s) = s^i, \tag{4.17}$$

for a death process of quadratic type such that $h(s) = p_0 + p_1 s$ ($a_{i,i-2} = i(i-1)\lambda p_0$, $a_{i,i-1} = i(i-1)\lambda p_1$, $a_{ii} = -i(i-1)\lambda$) leads to an integral representation for $F_i(t;s)$ generalizing the formula (4.12) for a death process of linear type.

Below, for a function $H(v;s)$ of two variables we use the notation $[H(v;s)]'_v = \partial H(v;s)/\partial v$. We denote the imaginary unit by ω, $\omega^2 = -1$.

Theorem 4.1. *Let a Markov process on the state space* $\mathbb{N}$ *be given by the infinitesimal characteristics* (4.14), *and let* $h(s) = p_0 + p_1 s$. *The generating function of the transition probabilities is*

$$\begin{aligned} F_i(t;s) = 1 + \frac{e^{\lambda t/4}}{2\sqrt{2\pi\lambda t}} \int_{-\infty}^{\infty} e^{-v^2/(\lambda t)} \Bigg[\int_{\cos 2v}^{1} \frac{1}{\sqrt{y - \cos 2v}} \\ \times \left(\frac{1}{2\pi\omega} \oint_{0+} \left(\frac{\frac{1}{2}p_1(\eta - 1) - \frac{1}{2}(1+p_0)\sqrt{1 - 2\eta y + \eta^2} + s}{\eta} \right)^i \right. \\ \left. \times \frac{d\eta}{\sqrt{1 - 2\eta y + \eta^2}} \right) dy \Bigg]'_v dv, \quad i \in \mathbb{N}. \end{aligned} \tag{4.18}$$

Proof. Let us consider the partial differential equations (for $|s| < 1$)

$$\frac{\partial \mathcal{F}}{\partial t} = \lambda z^2 \left(p_0 \mathcal{F} + p_1 \frac{\partial \mathcal{F}}{\partial z} - \frac{\partial^2 \mathcal{F}}{\partial z^2} \right), \tag{4.19}$$

$$\frac{\partial \mathcal{F}}{\partial t} = \lambda (p_0 + p_1 s - s^2) \frac{\partial^2 \mathcal{F}}{\partial s^2}, \quad \mathcal{F}(0;z;s) = e^{zs}. \tag{4.20}$$

(a) *Method of separation of variables.* We seek a solution in the form of Fourier series

$$\mathcal{F}(t;z;s) = \sum_{n=0}^{\infty} A_n \widetilde{C}_n(z) C_n(s) e^{-\lambda_n t}. \tag{4.21}$$

Substituting (4.21) into the equations (4.19) and (4.20), we obtain equations for the functions $\widetilde{C}_n(z)$ and $C_n(s)$:

$$\lambda z^2 (p_0 \widetilde{C}_n(z) + p_1 \widetilde{C}'_n(z) - \widetilde{C}''_n(z)) + \lambda_n \widetilde{C}_n(z) = 0, \tag{4.22}$$

$$\lambda (p_0 + p_1 s - s^2) C''_n(s) + \lambda_n C_n(s) = 0, \quad n = 0, 1, \dots . \tag{4.23}$$

The differential equations (4.17) and (4.23) were studied in [88] for $p_0 = 0$ and for $p_0 = 1$. Following the lines of [88], one can show by using the assumptions about the Markov process under consideration that the equation (4.23) satisfies the boundary condition of the form "$C_n(s)$ is a polynomial." In this case, there is a sequence of 'eigenvalues' $\lambda_n = n(n-1)\lambda$, $n = 0, 1, \dots$ ([64], Part. II, Chap. 3, § 9.7), and corresponding to every λ_n is an 'eigenfunction'

$$C_n(s) = C_n^{-1/2} \left(\frac{2s - p_1}{1 + p_0} \right),$$

where $C_n^{-1/2}(s)$ are the Gegenbauer polynomials ([73], cf. (52.4) for $\sigma = -1/2$):

$$C_n^{-1/2}(s) = \sum_{k=0}^{[n/2]} \frac{(-1)^k(-1/2)(-1/2+1)\cdots(-1/2+n-k-1)}{k!\,(n-2k)!}(2s)^{n-2k}. \quad (4.24)$$

Accordingly, (4.22) becomes

$$z^2(p_0\widetilde{C}_n(z) + p_1\widetilde{C}_n'(z) - \widetilde{C}_n''(z)) + n(n-1)\widetilde{C}_n(z) = 0 \quad (4.25)$$

and can be reduced to the Bessel equation ([73], cf. (44.1) for $\alpha = -1/2$, $\beta = 1$, $\gamma^2 = -1$, and $\nu = n - 1/2$). It follows from the conditions on the Markov process that a solution of interest is analytic in the entire complex plane; following [73], we obtain

$$\widetilde{C}_n(z) = -\sqrt{\pi(1+p_0)z}\, e^{p_1 z/2} I_{n-1/2}((1+p_0)z/2),$$

where $I_{n-1/2}(z)$ are the modified Bessel functions. Thus, the desired series (4.21) is of the form

$$\mathcal{F}(t;z;s) = -\sqrt{\pi(1+p_0)z} \times \sum_{n=0}^{\infty} A_n e^{p_1 z/2} I_{n-1/2}\left(\frac{(1+p_0)z}{2}\right) C_n^{-1/2}\left(\frac{2s-p_1}{1+p_0}\right) e^{-n(n-1)\lambda t}.$$

The values of A_n can be determined by comparing the initial conditions $\mathcal{F}(0;z;s) = e^{zs}$ with the Sonin expansion for the exponential function (see [73], (53.2) and [100], the sum (5.13.3.3)),

$$e^{zs} = -\sqrt{2\pi z}\sum_{n=0}^{\infty}\left(n - \frac{1}{2}\right) I_{n-1/2}(z) C_n^{-1/2}(s).$$

Hence, $A_n = n - 1/2$, and we obtain the expression

$$\mathcal{F}(t;z;s) = -\sqrt{\pi(1+p_0)z} \times \sum_{n=0}^{\infty}\left(n - \frac{1}{2}\right) e^{p_1 z/2} I_{n-1/2}\left(\frac{(1+p_0)z}{2}\right) C_n^{-1/2}\left(\frac{2s-p_1}{1+p_0}\right) e^{-n(n-1)\lambda t}. \quad (4.26)$$

The convergence of the series (4.26) for any z and s and any $t \in [0,\infty)$ follows from the convergence of the Sonin expansion.

(b) *Integral representation.* Let us use the following formulae ([99], the integral (2.5.36.1), [100], the integral (2.17.1.7)):

$$e^{-n(n-1)\lambda t} = \frac{e^{\lambda t/4}}{\sqrt{\pi\lambda t}}\int_{-\infty}^{\infty} e^{-v^2/(\lambda t)}\cos(2n-1)v\,dv, \quad n = 0,1,\ldots,$$

$$\sin(2n-1)v = \frac{2n-1}{2\sqrt{2}}\int_{\cos 2v}^{1}\frac{P_{n-1}(y)\,dy}{\sqrt{y-\cos 2v}}, \quad n = 1,2,\ldots,$$

where the $P_n(y)$ are the Legendre polynomials. Differentiating the last expression, we obtain the following integral representation for the exponential function:

$$e^{-n(n-1)\lambda t} = \frac{e^{\lambda t/4}}{2\sqrt{2\pi\lambda t}} \int_{-\infty}^{\infty} e^{-v^2/(\lambda t)} \left[\int_{\cos 2v}^{1} \frac{P_{n-1}(y)\,dy}{\sqrt{y-\cos 2v}} \right]' dv, \quad n = 1, 2, \dots. \tag{4.27}$$

Let us consider another solution of (4.25),

$$\widetilde{D}_n(z) = \sqrt{(1+p_0)z/\pi}\, e^{p_1 z/2} K_{n-1/2}((1+p_0)z/2),$$

where $K_{n-1/2}(z)$ is the Macdonald function ([8], Chap. 7, § 2), that is,

$$\sqrt{\frac{2z}{\pi}}\, K_{n-1/2}(z) = e^{-z} \sum_{k=0}^{n-1} \frac{(n-1+k)!}{k!\,(n-1-k)!(2z)^k}. \tag{4.28}$$

We can verify the integral relation

$$C_n(s) = -\frac{2}{1+p_0}\, \frac{1}{2\pi\omega} \oint_{0+} \frac{e^{s\xi - p_1\xi}}{\xi^2} \widetilde{D}_n(\xi)\, d\xi, \qquad n = 2, 3, \dots, \tag{4.29}$$

where the integration contour goes around the point 0 in the complex plane in the positive direction, by straightforward calculations using the explicit expressions (4.24) and (4.28).

Let us substitute the integral (4.27) in the series (4.26) and replace $C_n(s)$ by the integral (4.29). One can readily justify changing the order of summation, integration, and differentiation, and hence

$$\begin{aligned}
\mathcal{F}(t;z;s) = {} & e^{p_1 z/2} \cosh \frac{(1+p_0)z}{2} + e^{p_1 z/2} \sinh \frac{(1+p_0)z}{2}\, \frac{2s-p_1}{1+p_0} \\
& + \frac{e^{\lambda t/4}}{2\sqrt{2\pi\lambda t}} \int_{-\infty}^{\infty} e^{-v^2/(\lambda t)} \Bigg[\int_{\cos 2v}^{1} \frac{1}{\sqrt{y-\cos 2v}} \Bigg(\frac{1}{2\pi\omega} \oint_{0+} \frac{e^{s\xi + p_1 z/2 - p_1\xi/2}}{\xi^2} \\
& \times \frac{2}{1+p_0} \sum_{n=2}^{\infty} \left(n - \frac{1}{2}\right) \sqrt{\pi(1+p_0)z}\, I_{n-1/2}\left(\frac{(1+p_0)z}{2}\right) \\
& \times \sqrt{\frac{(1+p_0)\xi}{\pi}}\, K_{n-1/2}\left(\frac{(1+p_0)\xi}{2}\right) P_{n-1}(y)\, d\xi \Bigg) dy \Bigg]'_v dv.
\end{aligned} \tag{4.30}$$

Up to the first summand, the series in (4.30) coincides with the Gegenbauer addition formula for cylindrical functions ([73], the last formula in § 53, [100], the sum (5.10.3.5)), namely, if $|z| < |\xi|$ and $|y| \leqslant 1$, then

$$\sum_{n=1}^{\infty} \left(n - \frac{1}{2}\right) \sqrt{2\pi z}\, I_{n-1/2}(z) \sqrt{\frac{2\xi}{\pi}}\, K_{n-1/2}(\xi) P_{n-1}(y) = \frac{z\xi e^{-\sqrt{z^2+\xi^2-2z\xi y}}}{\sqrt{z^2+\xi^2-2z\xi y}}. \tag{4.31}$$

Substituting the sum (4.31) into (4.30) and carrying out some manipulations related to the first summand of the series, we arrive at the integral representation

$$\begin{aligned}
\mathcal{F}(t;z;s) = {} & e^z + \frac{e^{\lambda t/4}}{2\sqrt{2\pi\lambda t}} \int_{-\infty}^{\infty} e^{-v^2/(\lambda t)} \Bigg[\int_{\cos 2v}^{1} \frac{1}{\sqrt{y-\cos 2v}} \\
& \times \left(\frac{1}{2\pi\omega} \oint_{0+} e^{s\xi + p_1 z/2 - p_1\xi/2 - (1+p_0)\sqrt{z^2+\xi^2-2z\xi y}/2} \frac{z\,d\xi}{\xi\sqrt{z^2+\xi^2-2z\xi y}} \right) dy \Bigg]'_v dv.
\end{aligned}$$

Let us make the change of variable $\xi = z/\eta$ in the third integral; we thus obtain the following expression for the double generating function:

$$\mathcal{F}(t;z;s) = e^z + \frac{e^{\lambda t/4}}{2\sqrt{2\pi\lambda t}} \int_{-\infty}^{\infty} e^{-v^2/(\lambda t)} \Bigg[\int_{\cos 2v}^{1} \frac{1}{\sqrt{y-\cos 2v}} \\ \times \left(\frac{1}{2\pi\omega} \oint_{0+} e^{z\eta^{-1}(s+p_1(\eta-1)/2-(1+p_0)\sqrt{\eta^2+1-2\eta y}/2)} \frac{d\eta}{\sqrt{\eta^2+1-2\eta y}} \right) dy \Bigg]'_v dv.$$

The formula (4.18) follows from the definition (4.3) and the expansion $e^{zx} = \sum_{i=0}^{\infty}(z^i/i!)\,x^i$ by equating the coefficients of like powers of z. This proves Theorem 4.1.

4.2.2. Third equation for a death process of quadratic type. The integral with respect to η in (4.18) can be evaluated by using the generating function of the Legendre polynomials

$$P_n(y) = \frac{1}{2\pi\omega} \oint_{0+} \frac{1}{\sqrt{1-2\eta y+\eta^2}} \frac{d\eta}{\eta^{n+1}}, \qquad n = 0, 1, \dots. \tag{4.32}$$

The Legendre polynomials have the following integral representation as well ([34], Chap. 5, § 24, (109)):

$$P_n(y) = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\psi}{(y+\omega\sqrt{1-y^2}\,\cos\psi)^{n+1}}, \qquad n = 0, 1, \dots.$$

We make the change of variable $x = \omega(\psi + \pi/2)$ in the last integral. This gives

$$P_n(y) = \frac{1}{2\pi\omega} \int_T \frac{dx}{(y+\sqrt{1-y^2}\sinh x)^{n+1}}, \qquad n = 0, 1, \dots, \tag{4.33}$$

where $T = \{x = \omega u,\ \pi/2 \leqslant u \leqslant 5\pi/2\}$ is a segment in the complex plane oriented in increasing order of u. The representation (4.32) for $P_n(y)$ is related to the integral (4.33) by the change of variable $\eta = (\sqrt{1-y^2}e^x + 2y - \sqrt{1-y^2}e^{-x})/2$ (integral representations for the Legendre polynomials and the corresponding changes of variables are considered in [34], Chap. 1, § 8.19).

In the integral (4.18) we make the above change of variable η. The *generalized branching property* for a death process of quadratic type is

$$F_i(t;s) = \frac{e^{\lambda t/4}}{2\sqrt{2\pi\lambda t}} \int_{-\infty}^{\infty} e^{-v^2/(\lambda t)} \\ \times \left[\int_{\cos 2v}^{1} \frac{1}{\sqrt{y-\cos 2v}} \left(\frac{1}{2\pi\omega} \int_T \varphi^i(x,y;s)\,dx \right) dy \right]'_v dv, \tag{4.34}$$

where the function φ is linear in the variable s,

$$\varphi(x,y;s) = \frac{-\frac{1}{2}\big(p_0\sqrt{1-y^2}\,e^x + p_1(1-y) + \sqrt{1-y^2}\,e^{-x}\big) + s}{\frac{1}{2}\big(\sqrt{1-y^2}\,e^x + 2y - \sqrt{1-y^2}\,e^{-x}\big)}.$$

Substituting the double generating function

$$\mathcal{F}(t;z;s) = \frac{e^{\lambda t/4}}{2\sqrt{2\pi\lambda t}} \int_{-\infty}^{\infty} e^{-v^2/(\lambda t)} \times \left[\int_{\cos 2v}^{1} \frac{1}{\sqrt{y-\cos 2v}} \left(\frac{1}{2\pi\omega} \int_T e^{z\varphi(x,y;s)}\, dx \right) dy \right]'_v dv$$

in the equation (4.16), we obtain a non-linear first-order partial differential equation for the generating function $\varphi(x,y;s)$:

$$(h(s)-s^2)\left(\frac{\partial\varphi}{\partial s}\right)^2 - (h(\varphi)-\varphi^2) = (1+p_0)\frac{\partial\varphi}{\partial x}\,, \tag{4.35}$$

where $h(s) = p_0 + p_1 s$.

4.2.3. Third equation for a birth process of quadratic type. For a birth process in which $h(s) = s^3$ ($a_{ii} = -i(i-1)\lambda$, $a_{i,i+1} = i(i-1)\lambda$), the construction of the exact closed solution of the equation

$$\frac{\partial F_i(t;s)}{\partial t} = \lambda(s^3 - s^2)\frac{\partial^2 F_i(t;s)}{\partial s^2}\,, \qquad F_i(0;s) = s^i,$$

leads to an integral representation for $F_i(t;s)$ generalizing the above formula (4.13) for a birth process of linear type.

Theorem 4.2. *Let a Markov process on the state space $\mathbb{N}$ be given by the infinitesimal characteristics* (4.14), *and let $h(s) = s^3$. The generating function of the transition probabilities has the form* (*with $F_0(t;s) \equiv 1$*)

$$F_i(t;s) = \frac{e^{\lambda t/4}}{2\sqrt{2\pi\lambda t}} \int_{-\infty}^{\infty} e^{-v^2/(\lambda t)} \times \left[\int_{\cos 2v}^{1} \frac{1}{\sqrt{y-\cos 2v}} \left(\frac{1}{2\pi\omega} \int_T s\varphi^{i-1}(x,y;s)\, dx \right) dy \right]'_v dv, \tag{4.36}$$

$i = 1, 2, \ldots,$ *where $T = \{x = \omega u,\ \pi/2 \leqslant u \leqslant 5\pi/2\}$ is a segment on the complex plane oriented in increasing order of u and the function φ is linear-fractional with respect to the variable s,*

$$\varphi(x,y;s) = \frac{s\frac{1}{2}\left(\sqrt{1-y^2}\, e^x + 2y - \sqrt{1-y^2}\, e^{-x}\right)}{1 - s\frac{1}{2}\left(-\sqrt{1-y^2}\, e^x + 1 - y\right)}\,.$$

Proof. In the quadratic case the expressions (2.9) for the transition probabilities of a Markov pure birth process become $P_{0j}(t) = \delta_j^0$, $P_{1j}(t) = \delta_j^1$, $P_{ij}(t) = 0$ for $i > j$, and

$$P_{ij}(t) = \frac{(j-1)!\,(j-2)!}{(i-1)!\,(i-2)!} \times \sum_{n=i}^{j} \frac{(-1)^{n-i}(2n-1)(n+i-2)!}{(j-n)!\,(j+n-1)!\,(n-i)!} e^{-n(n-1)\lambda t}\,, \quad 2 \leqslant i \leqslant j. \tag{4.37}$$

The definition (4.1) of the generating function $\mathcal{F}_i(t;s)$ and the formula (4.36) imply the integral representation (for $2 \leqslant i \leqslant j$)

$$P_{ij}(t) = \frac{e^{\lambda t/4}}{2\sqrt{2\pi\lambda t}} \int_{-\infty}^{\infty} e^{-v^2/(\lambda t)} \biggl[\int_{\cos 2v}^{1} \frac{1}{\sqrt{y - \cos 2v}} \biggl(\frac{1}{2\pi\omega} \int_T \binom{j-2}{j-i} \times \biggl(\frac{1}{2}(\sqrt{1-y^2}e^x + 2y - \sqrt{1-y^2}e^{-x}) \biggr)^{i-1} \times \biggl(\frac{1}{2}(-\sqrt{1-y^2}e^x + 1 - y) \biggr)^{j-i} dx \biggr) dy \biggr]'_v dv.$$

The proof of the theorem amounts to computing the triple integral, which leads to the expressions (4.37). The integral with respect to x is immediate and gives a linear combination of Legendre polynomials $P_n(y)$; next we use the integral (4.27). For $j = i, i+1, i+2$ some easy manipulations lead to the following expressions for the transition probabilities:

$$P_{ii}(t) = e^{-i(i-1)\lambda t}, \quad P_{i,i+1}(t) = \frac{i-1}{2}\bigl(e^{-i(i-1)\lambda t} - e^{-(i+1)i\lambda t}\bigr),$$
$$P_{i,i+2}(t) = \frac{i(i+1)}{4(2i+1)}\bigl((i+1)e^{-i(i-1)\lambda t} - (2i+1)e^{-(i+1)i\lambda t} + ie^{-(i+2)(i+1)\lambda t}\bigr).$$

We then use induction on j. This completes the proof of Theorem 4.2.

An equation for the generating function $\varphi(x,y;s)$ follows from the generalized branching property (4.36). We substitute the double generating function

$$\mathcal{F}(t;z;s) = 1 + \frac{e^{\lambda t/4}}{2\sqrt{2\pi\lambda t}} \int_{-\infty}^{\infty} e^{-v^2/(\lambda t)} \times \biggl[\int_{\cos 2v}^{1} \frac{1}{\sqrt{y - \cos 2v}} \biggl(\frac{1}{2\pi\omega} \int_T \frac{s}{\varphi(x,y;s)} (e^{z\varphi(x,y;s)} - 1)\, dx \biggr) dy \biggr]'_v dv$$

in the equation (4.16) and obtain the third equation

$$(h(s) - s^2)\biggl(\frac{\partial\varphi}{\partial s}\biggr)^2 - (h(\varphi) - \varphi^2) = \varphi^2 \frac{\partial\varphi}{\partial x}, \tag{4.38}$$

where $h(s) = s^3$. For the partial differential equations (4.35) and (4.38) we can find the complete integrals by standard methods [63].

§ 4.3. The branching property and the Green function for parabolic equations

In § 4.2 we obtained the non-linear equations (4.35) (for a death process) and (4.38) (for a birth process), and they differ by a factor on the right-hand side. This difference is eliminated by studying the unifying case, that is, a birth-and-death Markov process of quadratic type for which $h(s) = p_0 + p_1 s + p_3 s^3$. To derive the

third equation, it is necessary to reveal the generalized branching property, that is, to find the exact solution of the parabolic differential equation

$$\frac{\partial F_i(t;s)}{\partial t} = \lambda(p_0 + p_1 s + p_3 s^3 - s^2)\frac{\partial^2 F_i(t;s)}{\partial s^2}, \qquad F_i(0;s) = s^i, \tag{4.39}$$

in a form similar to the branching property (2.11) (the desired integral representation of the solution contains a linear-fractional function of the variable s).

In a cycle of papers in 1982–1995 (see [80]–[82] and other papers), Letessier and Valent obtained solutions (in the form of series in special functions) of the second Kolmogorov equation for certain birth-and-death processes of quadratic, cubic, and biquadratic types by using the method of separation of variables. Applying the methods of [120], one can show that a solution of (4.39) having probabilistic meaning (that is, analytic in the circle $|s| < 1$) can be represented by a Fourier series

$$F_i(t;s) = \sum_{n=0}^{\infty} A_n C_n(s) e^{-\lambda_n t}, \qquad |s| < 1, \tag{4.40}$$

where the eigenvalues are $\lambda_n = n(n-1)\lambda K$ and the constant $K > 0$ can be expressed in terms of the elliptic integral

$$\int \frac{ds}{\sqrt{p_0 + p_1 s + p_3 s^3 - s^2}}\,.$$

The equation

$$(p_0 + p_1 s + p_3 s^3 - s^2)C_n''(s) + \lambda_n C_n(s) = 0 \tag{4.41}$$

for the eigenfunction $C_n(s)$ belongs to the class of Heun equations (a second-order Fuchsian equation with four singular points ([9], Chap. 15, § 3).

The equation (4.23) (for the function $C_n(s)$) studied in 4.2.1 belongs to the class of hypergeometric equations (a second-order Fuchsian equation with three singular points ([7], Chap. 2, § 6). The construction in 4.2.1 of a closed solution of the form (4.18) for the parabolic equation (4.17) is contingent on the consideration of functions studied in detail in the theory of special functions, namely, the integral relation between hypergeometric functions ([101], Chap. 7) and the Gegenbauer addition theorem were applied.

In the case of (4.41), for special functions of the Heun class there are no known integral relations generalizing those for hypergeometric functions; no addition theorems have been obtained for Heun functions (see the survey in [112]). It is unclear how to sum the series (4.40) and how to construct a closed solution of the equation (4.39). The desired integral representation for $F_i(t;s)$ contains elliptic functions.

The Green function. Let us discuss the integral representations of solutions found in Theorems 4.1 and 4.2 from the point of view of the general theory of partial differential equations. Suppose that by using the method of Green functions for equations of parabolic type ([2], Chap. 6, § 3.2) we have constructed a solution of the problem (4.39),

$$F_i(t;s) = \int_{-\infty}^{\infty} G(t;\xi,s)\,\xi^i\,d\xi, \tag{4.42}$$

where

$$G(t;\xi,s)=\sum_{n=0}^{\infty}\frac{C_n(\xi)C_n(s)}{\|C_n\|^2}e^{-\lambda_n t} \tag{4.43}$$

is the Green function; the norm $\|C_n\|$ is taken in the corresponding function space ([2], [120]).

In the proofs of Theorems 4.1 and 4.2 we solved the problem of summing the series (4.43), and a non-closed representation of the Green function was reduced to a closed integral expression. After substituting the integral thus obtained in (4.42) and making the change of variable $\xi=\varphi(\bar{x};s)$, the formula (4.42) becomes

$$F_i(t;s)=\int\cdots\int_{\widetilde{T}}\widetilde{G}(t;\bar{x})\varphi^i(\bar{x};s)\,d\bar{x}. \tag{4.44}$$

Here a change of variable is possible for which the function φ in the representation (4.44) is linear or linear-fractional with respect to the variable s, and the multiple integral is an expectation (a probabilistic interpretation of the formula (4.44) is given in §§ 5.3, 5.4). Thus, the closed solutions found in Theorems 4.1 and 4.2 are special cases of the formula (4.42). We also note that closed expressions for the Green functions are known for parabolic equations with constant coefficients [2].

4.3.1. Open problems. Below we present Kolmogorov linear equations for some cases in which the methods of Chapter 4 can be used to prove an integral representation of the solution and to derive a non-linear equation similar to the equations (4.35) and (4.38). The problems are as follows.

1. Obtain the third equation for a Markov death process on the state space $\mathbb{N}$ for which the first and second equations for the double generating functions have the form [62] (with $\lambda\geqslant 0$ and $\mu\geqslant 0$)

$$\begin{aligned}\frac{\partial\mathcal{F}}{\partial t}&=\lambda z^2\left(p_0\mathcal{F}+p_1\frac{\partial\mathcal{F}}{\partial z}-\frac{\partial^2\mathcal{F}}{\partial z^2}\right)+\mu z\left(\mathcal{F}-\frac{\partial\mathcal{F}}{\partial z}\right),\\ \frac{\partial\mathcal{F}}{\partial t}&=\lambda(p_0+p_1s-s^2)\frac{\partial^2\mathcal{F}}{\partial s^2}+\mu(1-s)\frac{\partial\mathcal{F}}{\partial s}\,,\quad \mathcal{F}(0;z;s)=e^{zs}.\end{aligned}$$

2. Obtain the third equation for a process with immigration of particles of class B_2 for which the Kolmogorov equations are

$$\begin{aligned}\frac{\partial\mathcal{F}}{\partial t}&=\lambda z^2\left(\mathcal{F}-\frac{\partial^2\mathcal{F}}{\partial z^2}\right)+\mu\left(\frac{\partial^2\mathcal{F}}{\partial z^2}-\mathcal{F}\right),\\ \frac{\partial\mathcal{F}}{\partial t}&=\lambda(1-s^2)\frac{\partial^2\mathcal{F}}{\partial s^2}+\mu(s^2-1)\mathcal{F},\quad \mathcal{F}(0;z;s)=e^{zs}.\end{aligned}$$

Consider the interaction schemes $2T_1\to k_1T_1$ for $k_1=0,1$ and $0\to k_0T_1$ for $k_0=1,2$.

3. Obtain the third equation for a process with particles of final type such that the equation for the double generating function is of the form [60] (for $\lambda > 0$)

$$\frac{\partial \mathcal{F}}{\partial t} = \lambda z_1^2 \left(h_0\left(\frac{\partial}{\partial z_2}\right) + \frac{\partial}{\partial z_1} h_1\left(\frac{\partial}{\partial z_2}\right) + \frac{\partial^2}{\partial z_1^2} h_2\left(\frac{\partial}{\partial z_2}\right) - \frac{\partial^2}{\partial z_1^2}\right)\mathcal{F},$$

$$\frac{\partial \mathcal{F}}{\partial t} = \lambda(h_0(s_2) + s_1 h_1(s_2) + s_1^2 h_2(s_2) - s_1^2)\frac{\partial^2 \mathcal{F}}{\partial s_1^2}, \quad \mathcal{F}(0; z; s) = e^{zs}.$$

Consider a more general scheme $2T_1 \to \gamma_1 T_1 + \gamma_2 T_2$, $\gamma_1 = 0, 1, 2, 3$.

4. Obtain the third equation for a process in 2.2.2 with interaction of particles of different types, $T_1 + T_2 \to T_3$; the corresponding linear equations are

$$\frac{\partial \mathcal{F}}{\partial t} = \lambda z_1 z_2 \left(\frac{\partial \mathcal{F}}{\partial z_3} - \frac{\partial^2 \mathcal{F}}{\partial z_1 \partial z_2}\right), \quad \frac{\partial \mathcal{F}}{\partial t} = \lambda(s_3 - s_1 s_2)\frac{\partial^2 \mathcal{F}}{\partial s_1 \partial s_2}, \quad \mathcal{F}(0; z; s) = e^{zs}.$$

5. The first and the second equations for the double generating function of transition probabilities for a simple death process of polynomial type are as follows (for $k = 3, 4, \dots$):

$$\frac{\partial \mathcal{F}}{\partial t} = \lambda z^k \left(\frac{\partial^{k-1} \mathcal{F}}{\partial z^{k-1}} - \frac{\partial^k \mathcal{F}}{\partial z^k}\right), \quad \frac{\partial \mathcal{F}}{\partial t} = \lambda(s^{k-1} - s^k)\frac{\partial^k \mathcal{F}}{\partial s^k}, \quad \mathcal{F}(0; z; s) = e^{zs}.$$

Find the third equation. Obtain the third equation for an arbitrary simple death process of class B_2.

Obtain the third equation for a pure birth process of class B_2.

CHAPTER V

THE IDENTITY PRINCIPLE FOR PARTICLES AND INDEPENDENCE CONDITIONS

The definition and the equations of Markov processes with interaction are connected with the conditions of non-equilibrium statistical physics. In § 5.1 it is shown that the first Kolmogorov system of differential equations for a process in M_1 is a chain of equations for α-particle distribution functions $\{P_{\alpha\beta}(t), \beta \in \mathbb{N}^n\}$. In § 5.2 we discuss the applicability of the identity principle for particles and the de Finetti–Khinchin symmetry theorem to the derivation of the kinetic equation for the one-particle distribution function, and we analyze the conditions under which the mathematical description of non-equilibrium states of physical systems of interacting particles can be reduced to the consideration of the kinetic equation. In § 5.3 we take a model of a bimolecular reaction in the form of a Markov process for a stochastic system of interacting particles. The state space of particle trajectories corresponding to a stochastic process with interaction is considered, and the kinetic equation is obtained by transforming the state space of particle trajectories to a set of forests.

§ 5.1. The Bogolyubov chain of equations for a Markov system of interacting particles

In the study of systems of interacting particles in non-equilibrium statistical physics, one uses i-particle distribution functions $F_i(t; x_1, \dots, x_i)$ that describe the arrangement of i particles of the same type in the state space $(-\infty, \infty)$ at a time $t \in [0, \infty)$. Under general assumptions, a *chain of equations* was obtained

in [13] for such distribution functions,

$$\frac{\partial F_i}{\partial t} = \Phi_i(x_1, \dots, x_i; F_i) + \widetilde{\Phi}_{i+1}(x_1, \dots, x_{i+1}; F_{i+1}), \quad i = 1, 2, \dots, \tag{5.1}$$

where $\Phi_i(x_1, \dots, x_i; F_i)$ and $\widetilde{\Phi}_{i+1}(x_1, \dots, x_{i+1}; F_{i+1})$ are certain interrelated functionals. In [13] it is indicated that the one-particle distribution function is of primary interest. It is assumed that this function $F_1(t, x_1)$ satisfies a *kinetic equation* of the form

$$\frac{\partial F_1}{\partial t} = A(x_1; F_1), \tag{5.2}$$

where $A(x_1; F_1)$ is some functional. For a known function $F_1(t; x_1)$ one can find expressions for the i-particle distribution functions $F_i(t; x_1, \dots, x_i)$ with the help of the chain of equations (5.1).

A vast literature is devoted to systems of equations of the form (5.1) and to kinetic equations of the form (5.2) (see [96], [86], [87], and elsewhere). Other approaches are also used in the study of systems of interacting particles, for instance, equations for two-particle distribution functions are used [91]. The Markov property is often introduced when describing systems with interaction with the help of distribution functions.

Markov processes with interaction. Let a process with interaction on the state space $\mathbb{N}^n$ be given by $\varepsilon^1, \{p^1_\gamma\}, \{\varphi^1_\alpha\}, \dots, \varepsilon^l, \{p^l_\gamma\}, \{\varphi^l_\alpha\}$. The transition probabilities $\{P_{\alpha\beta}(t),\ \beta \in \mathbb{N}^n\}$ determine the α-particle distribution function. The multivariate generating function $F_\alpha(t; s)$ of the transition probabilities is a convolution of the α-particle distribution function. Let the conditions of § 1.1.1 hold.

Proposition 5.1. *The first system of Kolmogorov differential equations for the transition probabilities of a Markov process in M_1 can be represented in the form of the following chain of equations*:

$$\frac{\partial F_\alpha(t; s)}{\partial t} = \sum_{i=1}^{l} \varphi^i_\alpha \Big(-F_\alpha(t; s) + \sum_\gamma p^i_\gamma F_{\alpha - \varepsilon^i + \gamma}(t; s) \Big), \quad \alpha \in \mathbb{N}^n. \tag{5.3}$$

Proof. Let us use the definition of generating function in (1.10), the system of equations (1.2), and the definition (1.8) of the infinitesimal characteristics of a Markov processes with interaction:

$$\begin{aligned}
\frac{\partial F_\alpha(t; s)}{\partial t} &= \sum_\beta \frac{dP_{\alpha\beta}(t)}{dt} s^\beta = \sum_\beta \Big(\sum_\gamma a_{\alpha\gamma} P_{\gamma\beta}(t) \Big) s^\beta \\
&= \sum_\beta \Big(\sum_{i=1}^{l} \sum_{\gamma - \alpha + \varepsilon^i \geqslant 0} \varphi^i_\alpha p^i_{\gamma - \alpha + \varepsilon^i} P_{\gamma\beta}(t) - \sum_{i=1}^{l} \varphi^i_\alpha P_{\alpha\beta}(t) \Big) s^\beta \\
&= \sum_{i=1}^{l} \varphi^i_\alpha \Big(\sum_{\gamma - \alpha + \varepsilon^i \geqslant 0} p^i_{\gamma - \alpha + \varepsilon^i} \sum_\beta P_{\gamma\beta}(t) s^\beta - \sum_\beta P_{\alpha\beta}(t) s^\beta \Big) \\
&= \sum_{i=1}^{l} \varphi^i_\alpha \Big(\sum_\gamma p^i_\gamma F_{\alpha - \varepsilon^i + \gamma}(t; s) - F_\alpha(t; s) \Big).
\end{aligned}$$

This proves Proposition 5.1.

If the chain of equations (5.3) holds for a Markov process, then one can pose the problem of deriving an equation of the form (5.2) for this Markov process. For processes of class $B_1 \subset M_1$ the system of non-linear equations (1.22) for one-particle generating functions is of the form (5.2).

§ 5.2. The de Finetti–Khinchin symmetry theorem and the kinetic equation

Definition 5.2. Random variables $\xi_1, \xi_2, \dots, \xi_i$ are said to be *symmetrically dependent* (*permutable*) if all $i!$ permutations of the sequence $(\xi_1, \xi_2, \dots, \xi_i)$ have identical joint distributions, that is,

$$\mathsf{P}\{\xi_1 \leqslant x_1,\, \xi_2 \leqslant x_2,\, \dots,\, \xi_i \leqslant x_i\} = \mathsf{P}\{\xi_{j_1} \leqslant x_1,\, \xi_{j_2} \leqslant x_2,\, \dots,\, \xi_{j_i} \leqslant x_i\}$$

for all $i!$ permutations $\begin{pmatrix} 1, & 2, & \dots, & i \\ j_1, & j_2, & \dots, & j_i \end{pmatrix}$ and for any $x_1, x_2, \dots, x_i$.

We say that variables $\xi_1, \xi_2, \dots, \xi_i, \dots$ form an infinite sequence of symmetrically dependent random variables if $\xi_1, \xi_2, \dots, \xi_i$ are symmetrically dependent random variables for all $i = 2, 3, \dots$.

Theorem 5.3 (B. de Finetti and A. Ya. Khinchin ([66], [117], [83])). *Let* $(\Omega, \mathcal{A}, \mathsf{P})$ *be a probability space, and let* $\xi_1(\omega), \xi_2(\omega), \dots, \xi_i(\omega), \dots$ *be an infinite sequence of symmetrically dependent random variables. Let*

$$F(x_1, x_2, \dots, x_i) = \mathsf{P}\{\xi_1 \leqslant x_1, \xi_2 \leqslant x_2, \dots, \xi_i \leqslant x_i\}.$$

Then there is a random variable $\theta(\omega)$ *such that if*

$$\mathsf{P}\{\xi_1 \leqslant x \mid \theta = y\} = \varphi(x \mid y), \qquad \mathsf{P}\{\theta \leqslant x\} = H(x),$$

then

$$F(x_1, x_2, \dots, x_i) = \int_{-\infty}^{\infty} \varphi(x_1 \mid y)\varphi(x_2 \mid y) \cdots \varphi(x_i \mid y)\, dH(y), \quad i = 1, 2, \dots. \tag{5.4}$$

The integral (5.4) can be represented as the expectation [117]

$$\mathsf{P}\{\xi_1 \leqslant x_1,\, \xi_2 \leqslant x_2,\, \dots,\, \xi_i \leqslant x_i\} = \mathsf{E}\big(\mathsf{P}\{\xi_1 \leqslant x_1 \,|\, \theta\}\mathsf{P}\{\xi_2 \leqslant x_2 \,|\, \theta\} \cdots \mathsf{P}\{\xi_i \leqslant x_i \,|\, \theta\}\big).$$

Thus, the permutability property for an infinite sequence of symmetrically dependent random variables is equivalent to the property of conditional independence and conditional identical distribution. The proof of Theorem 5.3 gives no way to construct the relation (5.4). In general, the joint distribution function $F(x_1, x_2, \dots, x_i)$ of a finite sequence of random variables $\xi_1, \xi_2, \dots, \xi_i$ cannot be represented in the form (5.4).

The i-particle distribution functions $F_i(t; x_1, x_2, \dots, x_i)$ considered in § 5.1 are symmetric functions of the variables $x_1, x_2, \dots, x_i$ ([13], Chap. 2, § 6) because

microparticles of the same type are indistinguishable [36]. As applied to the i-particle functions, the integral representation (5.4) becomes

$$F_i(t; x_1, x_2, \dots, x_i) = \int_{-\infty}^{\infty} \varphi(x_1 \mid y)\varphi(x_2 \mid y) \cdots \varphi(x_i \mid y)\, dH(t; y), \quad i = 1, 2, \dots, \tag{5.5}$$

and the problem of deriving the kinetic equation reduces to the search for an averaging measure $H(t; y)$ and to the derivation of an equation for the one-particle conditional distribution function $\varphi(x \mid y)$ [49].

Independence conditions. The symmetry theorem can be applied to systems of interacting particles only if the definition of these systems is rigorous from the viewpoint of the theory of stochastic processes, that is, the *probability space* $(\Omega, \mathcal{A}, \mathsf{P})$ must be defined. In physical kinetics there are some equations that are used but not rigorously defined [79]. The equation of a diffusion process [69] and the equation of a branching process ([71], [106]) were the first kinetic equations studied in the theory of Markov processes. Both diffusion and branching processes correspond to a degeneration of the formula (5.5) for which the averaging measure is concentrated at a single point,

$$F_i(t; x_1, x_2, \dots, x_i) = F_1(t; x_1)F_1(t; x_2) \cdots F_1(t; x_i), \qquad i = 1, 2, \dots,$$

that is, the evolutions of individual particles in these processes are independent of one another. For a Markov branching process the system of differential equations (1.22) was obtained for one-particle generating functions in [71] on the basis of the assumption (1.21) that the evolutions of particles are independent.

In the mathematical description of physical processes one makes definite assumptions on the character of interaction (dependence) between the particles. In the probabilistic description of such processes, the main assumptions have to do with the mutual independence of various events. "In a sense, the notion of *independence* $\langle \dots \rangle$ occupies the central place in probability theory. $\langle \dots \rangle$ Accordingly, one of the main problems of the philosophy of science $\langle \dots \rangle$ is the clarification and refinement of premises under which given actual phenomena can be treated as independent" [70]. The following *independence conditions* are used in statistical physics when describing systems of interacting particles: the Markov property, the condition that the correlation between the positions of microparticles weakens as the distance between them increases, that is,

$$F_i(t; x_1, x_2, \dots, x_i) - F_1(t; x_1)F_1(t; x_2) \cdots F_1(t; x_i) \to 0$$

as all distances tend to infinity, $|x_k - x_l| \to \infty$ ([13], Chap. 1, § 1), and the independent process of collision acts of microparticles in a rarefied gas. The last condition is used in the Boltzmann kinetic equation ([79], [102]).

The set M_1 of Markov processes with interaction (defined in § 1.3) can be distinguished in the set M of all Markov processes by the following independence condition: *the result of interaction of a complex S_{ε^i}, $\varepsilon^i \in A$, of particles is independent of the presence of the other particles.* As applied to processes with discrete states, this condition was introduced in [79] for bimolecular processes with $|\varepsilon^i| = 2$ for $\varepsilon^i \in A$, and in [104] for branching processes with interaction.

We assume that for Markov processes with interaction the α-particle distribution functions $\{P_{\alpha\beta}(t),\ \beta \in \mathbb{N}^n\}$ have symmetry properties, because in the definition of these processes we introduced the identity condition for all particles of the same type.

§ 5.3. Transformation of the state space of particle trajectories for a system with interaction to a set of trees

Let us consider a process with interaction of class B_2 with the scheme $2T \to kT$, namely, a homogeneous Markov process ξ_t, $t \in [0, \infty)$, on the state space $\mathbb{N}$ determined by $\varepsilon = 2$, $\{p_k,\ k \in \mathbb{N};\ p_2 = 0\}$, $\{\varphi_i = i(i-1)\lambda,\ i \in \mathbb{N},\ \lambda > 0\}$. The first system of Kolmogorov differential equations can be represented in the form of a chain of equations

$$\frac{\partial F_i(t;s)}{\partial t} = -i(i-1)\lambda F_i(t;s) + i(i-1)\lambda \sum_{k=0}^{\infty} p_k F_{i-2+k}(t;s), \qquad i = 0, 1, \ldots;$$

the generating function $F_i(t;s)$ is a convolution of the distribution $\{P_{ij}(t),\ j \in \mathbb{N}\}$.

We interpret a state i of the process ξ_t as the presence of i particles of type T in some physical system. In a random time τ_i with $\mathsf{P}\{\tau_i \leqslant t\} = 1 - e^{-i(i-1)\lambda t}$ an interaction of some pair among i particles of the system occurs. This complex of two particles is transformed into an aggregate of k particles (descendants) of type T with the probability distribution $\{p_k\}$, corresponding to the passage of the Markov process ξ_t from the state i to the state $i-2+k$; the further evolution is similar. To describe this system of interacting particles mathematically, we can construct the space of elementary events that consists of a description of the evolution of each individual particle that ever existed in the system (the space of particle trajectories) and the corresponding probability measure (a measure on the particle trajectories). We denote this probability space by $(\Omega^{\mathrm{tr}}, \mathcal{A}^{\mathrm{tr}}, \mathsf{P}^{\mathrm{tr}})$. In Fig. 4 we show an example of a realization of a process with interaction of particles in the state space Ω^{tr}.

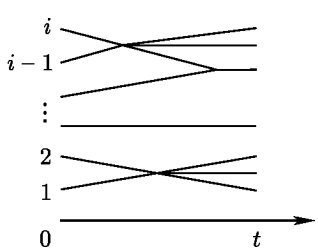

Figure 4. Realization of a stochastic process with interaction of particles

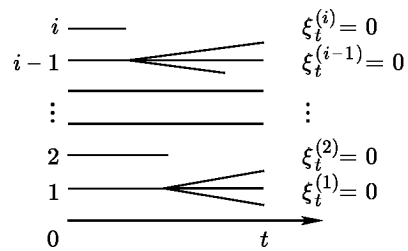

Figure 5. Transformation of the state space to a set of trees

Let us transform the state space of particle trajectories to a set of trees as follows: we assign the descendants of any pair of interacting particles to one of the particles in this pair. Then for each of the i initial particles we obtain randomly many descendants $\xi_t^{(l)}$ ($l = 1, \ldots, i$; see Fig. 5), and the following relation holds:

$$\xi_t = \xi_t^{(1)} + \cdots + \xi_t^{(i)}. \tag{5.6}$$

Since the particles are indistinguishable, we can assume that the random variables $\xi_t^{(l)}$ are permutable (symmetrically dependent). The equality (5.6) enables us to specify more precisely the assumption about the conditional independence and conditional identical distribution:

$$\begin{aligned} F_i(t;s) &= \sum_{j=0}^{\infty} P_{ij}(t)s^j = \mathsf{E}(s^{\xi_t} \mid \xi_0 = i) = \mathsf{E}s^{\xi_t^{(1)}+\dots+\xi_t^{(i)}} = \mathsf{E}\big(\mathsf{E}(s^{\xi_t^{(1)}+\dots+\xi_t^{(i)}} \mid \theta_t)\big) \\ &= \mathsf{E}\big(\mathsf{E}(s^{\xi_t^{(1)}} \mid \theta_t)\cdots\mathsf{E}(s^{\xi_t^{(i)}} \mid \theta_t)\big) = \mathsf{E}\big(\mathsf{E}(s^{\xi_t^{(1)}} \mid \theta_t)\big)^i \\ &= \int_{\{\omega\in\Omega^{\mathrm{tr}}\}} \varphi^i(\omega;s)\, H(t;d\omega), \end{aligned} \tag{5.7}$$

where $i \in \mathbb{N}$ and θ_t is a random variable on $(\Omega^{\mathrm{tr}}, \mathcal{A}^{\mathrm{tr}}, \mathsf{P}^{\mathrm{tr}})$ whose existence follows from Theorem 5.3. The form of the dependence of the function $\varphi(\omega;s)$ on the variable s is related to the structure of the trajectories (see § 5.4), and the function $H(t;d\omega)$ determines a random medium [1] corresponding to the Markov process in question. We note that the relation (5.6) coincides with the relation (4.10) for a Markov process of class B_1, and the integral representation (5.7) for the i-particle generating function generalizes the branching property (4.6) [15] for a process with independent particles.

A transformation of the state space of trajectories to a set of forests is possible if the equation (5.6) holds for an arbitrary system of interacting particles. In the case of continuous phase space the generating functionals [13], [106] are used to derive an integral representation of the form (5.7) for the i-particle distribution functions. The identity principle for particles [36] and symmetry properties are the most general aspects of importance in non-equilibrium statistical physics.

Kinetic equations. Finding an integral representation of the form (5.7) by probabilistic methods is difficult, because the trajectory space $(\Omega^{\mathrm{tr}}, \mathcal{A}^{\mathrm{tr}}, \mathsf{P}^{\mathrm{tr}})$ is complicated. The generalized branching property (5.7) for the generating function $F_i(t;s)$ of a Markov process ξ_t with pair interactions was revealed by analytic methods in Chapter 4, where closed solutions of the second Kolmogorov equation were constructed. In § 4.2.2 we found the averaging measure and the kinetic equation (4.35) for the one-particle generating function $\varphi(x,y;s)$ of the conditional transition probabilities for the bimolecular reaction $2T \to kT$, $k = 0, 1$. In § 4.2.3 we gave a similar non-linear first-order equation (4.38) for the function $\varphi(x,y;s)$ for the interaction scheme $2T \to 3T$. In accordance with Theorem 5.3, the multiple integral in the formulae (4.34) and (4.36) is treated as an expectation.

The problem of deriving the kinetic (third) equation for Markov processes with interaction can be solved by constructing exact solutions in the form of an integral representation (5.7) for the linear Kolmogorov equations in special cases. The examples of such solutions given in the present paper enable us to make conjectures about the types of non-linear equations for diverse classes of Markov processes: for B_1 this is a system of ordinary differential equations, for B_2 it is a first-order system of partial differential equations, for B_3 it is a system of integral equations of renewal type (for renewal equations, see [106], Chap. 8, § 7), and for the set M_1 it is a system of stochastic integral equations.

§ 5.4. Branching property for a simple death process

The simplest Markov process in the set M_1 is a death process ξ_t with $t \in [0, \infty)$ (on the state space $\mathbb{N}$) determined by $\varepsilon = 1$, $\{p_0 = 1\}$, $\{\varphi_0 = 0, \varphi_i > 0, i = 1, 2, \dots\}$. The Kolmogorov equations for the double generating function of the transition probabilities $\mathcal{F}(t; z; s)$ are of the form ((2.6) and (2.7))

$$\frac{\partial \mathcal{F}}{\partial t} = z(\mathcal{F} - D_z(\mathcal{F})), \quad \frac{\partial \mathcal{F}}{\partial t} = (1 - s)D_s(\mathcal{F}), \quad \mathcal{F}(0; z; s) = e(zs). \tag{5.8}$$

The process ξ_t stays at the initial state i during a random time τ_i with distribution $\mathsf{P}\{\tau_i \leqslant t\} = 1 - e^{-\varphi_i t}$. At the time τ_i the process passes to the state $i - 1$, and so on. An example of a realization of the stochastic process ξ_t is shown in Fig. 6.

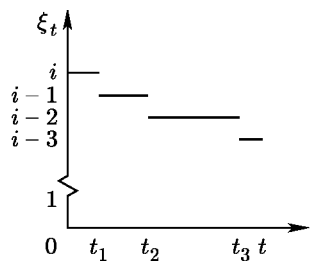

Figure 6. Death process for the state space $\mathbb{N}$

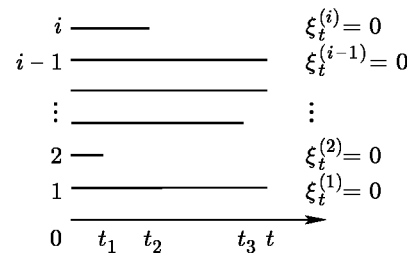

Figure 7. Death process for the state space Ω_{tr}

We interpret a state i of the process as the presence of i particles of the same type, and a transition of the process to the state $i - 1$ is the death of one of the particles; we have the scheme $T \to 0$. An example of a realization of the death process on the state space Ω^{tr} of particle trajectories is shown in Fig. 7. The random variables $\xi_t^{(l)}$ (the numbers of descendants for each of the initial particles, $l = 1, \dots, i$) take the values 0 and 1. Assuming that the random variables in the sum (5.6) are permutable, we see for an infinite sequence of symmetrically dependent random variables taking only the values 0 and 1 (cf. [26], vol. 2, Chap. 7, § 4) that

$$F_i(t; s) = \mathsf{E}(X_t + sY_t)^i, \qquad i \in \mathbb{N}, \tag{5.9}$$

where X_t and Y_t are some interacting stochastic processes.

Theorem 5.4. *Let a Markov death process satisfy the conditions $\varphi_{i+1} > \varphi_i$, $i \in \mathbb{N}$, and let $\lim_{i\to\infty} \varphi_i = \infty$. The double generating function of the transition probabilities can be represented by the Fourier series*

$$\mathcal{F}(t; z; s) = \sum_{n=0}^{\infty} \frac{1}{\varphi_1 \cdots \varphi_n} \widetilde{C}_n(z) C_n(s) e^{-\varphi_n t}, \tag{5.10}$$

where

$$\widetilde{C}_n(z) = z^n + \sum_{k=1}^{\infty} \frac{z^{n+k}}{(\varphi_{n+1} - \varphi_n) \cdots (\varphi_{n+k} - \varphi_n)},$$

$$C_n(s) = s^n + \sum_{k=0}^{n-1} \frac{\varphi_{k+1} \cdots \varphi_n}{(\varphi_k - \varphi_n) \cdots (\varphi_{n-1} - \varphi_n)} s^k.$$

The series (5.10) *is absolutely convergent for any* z, *any* s *with* $|s| < 1$, *and any* $t \in [0, \infty)$.

Proof. Let us use the definition of double generating function (2.5) and the explicit expressions (2.8) for the transition probabilities $P_{ij}(t)$ of the death process:

$$\begin{aligned}
\mathcal{F}(t; z; s) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{z^i}{\varphi_1 \cdots \varphi_i} P_{ij}(t) s^j \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{n=j}^{i} \frac{z^i}{\varphi_1 \cdots \varphi_j} \frac{e^{-\varphi_n t}}{(\varphi_i - \varphi_n)\cdots(\varphi_{n+1} - \varphi_n)(\varphi_{n-1} - \varphi_n)\cdots(\varphi_j - \varphi_n)} s^j \\
&= \sum_{n=0}^{\infty} \frac{e^{-\varphi_n t}}{\varphi_1 \cdots \varphi_n} \biggl(z^n + \sum_{i=n+1}^{\infty} \frac{z^i}{(\varphi_{n+1} - \varphi_n)\cdots(\varphi_i - \varphi_n)} \biggr) \\
&\qquad \times \biggl(s^n + \sum_{j=0}^{n-1} \frac{\varphi_{j+1} \cdots \varphi_n}{(\varphi_j - \varphi_n)\cdots(\varphi_{n-1} - \varphi_n)} s^j \biggr).
\end{aligned}$$

The convergence of the series for $\mathcal{F}(t; z; s)$ follows from the estimate

$$\biggl| \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{z^i}{\varphi_1 \cdots \varphi_i} P_{ij}(t) s^j \biggr| \leqslant \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{|z|^i}{\varphi_1 \cdots \varphi_i} |s|^j \leqslant \frac{1}{1 - |s|} \sum_{i=0}^{\infty} \frac{|z|^i}{\varphi_1 \cdots \varphi_i} < \infty,$$

for any z and any s with $|s| < 1$. This proves Theorem 5.4.

For a death process of linear type in which $\varphi_i = i\lambda$, the series (5.10) can readily be summed and leads to the branching property (4.12),

$$\mathcal{F}(t; z; s) = \sum_{n=0}^{\infty} \frac{(z/\lambda)^n}{n!} e^{z/\lambda} (s-1)^n \, e^{-n\lambda t} = e^{(z/\lambda)(1 + (s-1)e^{-\lambda t})}.$$

The sum of the series (5.10) was found in § 4.2.1 for a process of quadratic type such that $\varphi_i = i(i-1)\lambda$. Thus, the problem of justifying the probabilistic assumption that the generalized branching property (5.9) holds for the Markov death process reduces to the analytic problem of summing the series (5.10) under some assumptions about the function $\varphi_i = \varphi(i)$, $i \in \mathbb{N}$.

We note that the solution (5.10) of the equations (5.8) containing the Gel'fond–Leont'ev generalized differentiation operator has the form of a series of exponential functions [78]. For $t = 0$ we obtain an expansion of the eigenfunction (2.2):

$$e(zs) = \sum_{n=0}^{\infty} (\varphi_1 \cdots \varphi_n)^{-1} \widetilde{C}_n(z) C_n(s),$$

where the functions $\widetilde{C}_n(z)$ and $C_n(s)$ are related by an integral transform.

A pure birth process also belongs to the set M_1. As in the previous case, for the state space Ω^{tr} we can conjecture that the integral representation (5.7) of the i-particle generating function of the transition probabilities for this process has the form [53]

$$F_i(t; s) = \mathsf{E}\biggl(\frac{sY_t}{1 - sX_t} \biggr)^i, \qquad i \in \mathbb{N},$$

where X_t and Y_t are interconnected stochastic processes.

Theorem 5.5. *For a birth process of Poisson type the branching property holds in the following form (for $\omega^2 = -1$, $i = 2, 3, \dots$):*

$$F_i(t;s) = \int_0^1 \left(\frac{1}{2\pi\omega} \oint_{0+} \left[\int_0^\infty \left[\varphi^i(x,\eta,u;s) \frac{1+\eta}{1-x} \right]'_\eta d\eta \right]'_x \frac{e^{\lambda t(ux-1)}}{u}\, du \right) dx,$$

where φ is a linear-fractional function with respect to the variable s, namely,

$$\varphi(x,\eta,u;s) = \frac{s(1-x)/(1+\eta)}{1 - s(1/u + \eta(1-x)/(1+\eta))}\,.$$

Theorem 5.5 can be proved by a straightforward evaluation of the triple integral, which leads to the following expressions for the transition probabilities of the Poisson process: $P_{ij}(t) = 0$ for $0 \leqslant j < i$ and $P_{ij}(t) = e^{-\lambda t}(\lambda t)^{j-i}/(j-i)!$ for $j \geqslant i \geqslant 1$. The computation involves the Euler beta function ([44], [52]).

5.4.1. Open problems. The conjecture about the non-linear property (5.7) for Markov processes must be verified first of all for the cases in which the explicit expressions for the transition probabilities $P_{\alpha\beta}(t)$, $\alpha, \beta \in \mathbb{N}^n$, are either known or can be found for (at least) some values of α and β. The problems are as follows.

1. Discover the non-linear property of the transition probabilities (5.9) for a death process of power-law type for which $\varphi_i = i^\rho \lambda$, $0 < \rho < 1$ (for $\lambda > 0$).

2. The second Kolmogorov equation for the generating function of the transition probabilities of a two-dimensional death process ([76], [77]) of quadratic type is of the form

$$\frac{\partial F_\alpha(t;s)}{\partial t} = \lambda(p_{10}s_1 + p_{01}s_2 - s_1 s_2) \frac{\partial^2 F_\alpha(t;s)}{\partial s_1 \partial s_2}\,, \qquad F_\alpha(0;s) = s^\alpha.$$

Obtain the generalized branching property

$$F_{(\alpha_1,\alpha_2)}(t;s_1,s_2) = \mathsf{E}(X_t + s_1 Y_t)^{\alpha_1}(Z_t + s_2 U_t)^{\alpha_2}, \qquad (\alpha_1,\alpha_2) \in \mathbb{N}^2,$$

where X_t, Y_t, Z_t, U_t are interconnected stochastic processes. Obtain the property

$$F_{(\alpha_1,\alpha_2)}(t;s_1,s_2) = \mathsf{E}\left(\frac{s_1 Y_t}{1 - s_1 X_t}\right)^{\alpha_1} \left(\frac{s_2 U_t}{1 - s_2 Z_t}\right)^{\alpha_2}, \qquad (\alpha_1,\alpha_2) \in \mathbb{N}^2,$$

for a two-dimensional birth process for which the second equation is

$$\frac{\partial F_\alpha(t;s)}{\partial t} = \lambda(p_{21}s_1^2 s_2 + p_{12}s_1 s_2^2 - s_1 s_2) \frac{\partial^2 F_\alpha(t;s)}{\partial s_1 \partial s_2}\,, \qquad F_\alpha(0;s) = s^\alpha.$$

3. For a Markov birth-and-death process of Poisson type, obtain an integral representation for the i-particle generating function

$$F_i(t;s) = \mathsf{E}\left(\frac{X_t + sY_t}{Z_t + sU_t}\right)^i, \qquad i \in \mathbb{N},$$

where X_t, Y_t, Z_t, U_t are some stochastic processes, by using the explicit expressions (2.10) for the transition probabilities.

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