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FINAL PROBABILITIES FOR BECKER EPIDEMIC MARKOV PROCESS*

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(*Translated by the author*)

Abstract. We consider the Kolmogorov equations for transition probabilities if a three-dimensional Markov process is a special form. For a system of stationary first and second equations, an exact solution is obtain using the Fourier method. We obtain asymptotics for the expectation and variance of the final distribution and establish a limit theorem.

Key words. Becker epidemic process, Markov process, transition probabilities, equations for generating function, Fourier method, exact solution

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1. Double epidemic process.

On the set of states

$$N^3 = \{\alpha = (\alpha_1, \alpha_2, \alpha_3),\ \alpha_1, \alpha_2, \alpha_3 = 0, 1, 2, \dots\}$$

we consider a homogeneous (with respect to time) Markov process $\xi(t) = (\xi_1(t), \xi_2(t), \xi_3(t))$, $t \in [0, \infty)$, with transition probabilities $P^{(\alpha_1,\alpha_2,\alpha_3)}_{(\beta_1,\beta_2,\beta_3)}(t) = \mathbf{P}\{\xi(t) = (\beta_1, \beta_2, \beta_3) \mid \xi(0) = (\alpha_1, \alpha_2, \alpha_3)\}$. Suppose the transition probabilities have the following form as $\Delta t \to 0+$:

$$\begin{aligned}
&P^{(\alpha_1,\alpha_2,\alpha_3)}_{(\alpha_1,\alpha_2,\alpha_3-1)}(\Delta t) = (\mu_1\alpha_1\alpha_3 + \mu_2\alpha_2\alpha_3)\Delta t + o(\Delta t),\\
&P^{(\alpha_1,\alpha_2,\alpha_3)}_{(\alpha_1-1,\alpha_2,\alpha_3)}(\Delta t) = \rho_1\alpha_1\Delta t + o(\Delta t),\ P^{(\alpha_1,\alpha_2,\alpha_3)}_{(\alpha_1,\alpha_2-1,\alpha_3)}(\Delta t) = \rho_2\alpha_2\Delta t + o(\Delta t),\\
&P^{(\alpha_1,\alpha_2,\alpha_3)}_{(\alpha_1,\alpha_2,\alpha_3)}(\Delta t) = 1 - (\mu_1\alpha_1\alpha_3 + \mu_2\alpha_2\alpha_3 + \rho_1\alpha_1 + \rho_2\alpha_2)\Delta t + o(\Delta t),
\end{aligned}$$

where $\mu_1 > 0, \mu_2 > 0, \rho_1 > 0, \rho_2 > 0$.

The Markov process $\xi(t)$ can be interpreted as a model for a spread of two different epidemics through the population. The state of the process $(\alpha_1, \alpha_2, \alpha_3)$ means the presence of α_1 particles of type T_1 (the carriers of infection), α_2 particles of type T_2 (the carriers of another infection), and α_3 particles of type T_3 (the susceptible individuals). At a random time τ^1_α, $\mathbf{P}\{\tau^1_\alpha < t\} = 1 - e^{-\alpha_1\alpha_3\mu_1 t}$, a pair of particles of types T_1 and T_3 interacts and is transformed into a particle of type T_1. The process passes to the state $(\alpha_1, \alpha_2, \alpha_3 - 1)$. At a random time τ^2_α, $\mathbf{P}\{\tau^2_\alpha < t\} = 1 - e^{-\alpha_2\alpha_3\mu_2 t}$, a pair of particles of types T_2 and T_3 interacts and is transformed into a particle of type T_2. The process passes to the state $(\alpha_1, \alpha_2, \alpha_3 - 1)$. Besides, at a random time τ^3_α, $\mathbf{P}\{\tau^3_\alpha < t\} = 1 - e^{-\alpha_1\rho_1 t}$, a particle of type T_1 dies and the process passes to the state $(\alpha_1 - 1, \alpha_2, \alpha_3)$; at a random time τ^4_α, $\mathbf{P}\{\tau^4_\alpha < t\} = 1 - e^{-\alpha_2\rho_2 t}$, a particle of type T_2 dies and the process passes to the state $(\alpha_1, \alpha_2 - 1, \alpha_3)$. The random variables $\tau^1_\alpha, \tau^2_\alpha, \tau^3_\alpha, \tau^4_\alpha$ are independent and the process is in the state $(\alpha_1, \alpha_2, \alpha_3)$ during random time $\tau_\alpha = \min\{\tau^1_\alpha, \tau^2_\alpha, \tau^3_\alpha, \tau^4_\alpha\}$. We point out the absence of an interactions between particles of types T_1 and T_2.

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In the stochastic theory of epidemics the general stochastic epidemic (i.e., Bartlett–Mac-Kendric epidemic process [1]) and the simple stochastic epidemic (i.e., Weiss epidemic process [2]) are the basic models; both are considered on the set of states N^2. In the Bartlett–Mac-Kendric epidemic process an interacting pair of infected and susceptible particles is transformed into a pair of two infected particles (or two carriers). This process is rather complicated; some results were obtained by asymptotic methods (see [4], [5], etc.)

In the Weiss epidemic process an interacting pair of infected and susceptible particles is transformed into one carrier, i.e., one infected individual is removed from the population. This Markov process is more accesible to study; there are many generalizations of the Weiss model on the case of N^n. For instance, Gany introduced a carrier borne epidemic process with two stages of infection [13] on N^3 which is interpreted as a process in the spread of AIDS. In the present paper we consider the Markov process $\xi(t)$ which was introduced by Becker (see [3]) for $\rho_1 = \rho_2$ as a generalization of the Weiss epidemic process. In this double epidemic model $\xi(t)$ we assume that the population is under control, but carriers cannot be found according to the interpretation of particles of types T_1, T_2, T_3 given above.

2. The problem of final probabilities. For the process $\xi(t)$ we define *final probabilities* for absorbing states $(0,0,\gamma_3)$, $\gamma_3 = 0,1,2,\dots$ (i.e., all carriers of both types were removed)

$$(1)\qquad q_{(0,0,\gamma_3)}^{(\alpha_1,\alpha_2,\alpha_3)} = \lim_{t\to\infty} P_{(0,0,\gamma_3)}^{(\alpha_1,\alpha_2,\alpha_3)}(t),\ (\alpha_1,\alpha_2,\alpha_3)\in N^3;\quad \sum_{\gamma_3=0}^{\infty} q_{(0,0,\gamma_3)}^{(\alpha_1,\alpha_2,\alpha_3)} = 1.$$

The problem of determining the final probability distributions for a Markov process on N^2 was solved in [6] for a special case of a *branching process*, where the transition probabilities are nonlinearly related and the equation of the one-particle generation function is known [7]. The process $\xi(t)$ is a particular case of the class of Markov processes on N^n defined by Sevast'yanov in [8]. Calculation of the final distribution for the Markov process of this class [8] can be reduced to the solution of the stationary first (linear) Kolmogorov equation for the exponentional generating function of the transition probabilities (see [10, Chap. 3] and [9]).

Many papers are devoted to exact solutions for equations of Markov epidemic processes and to methods of their derivation (see [1], [2], [3], [4], etc). In the present paper, for the system consisting of the first and second Kolmogorov equations, we obtain the solution in the form of Fourier series. This solution for the exponentional (double) generating function for the transition probabilities is summed up to an integral form [10], [11], [12] in Theorem 1. In Theorem 2, the obtained generating function of the final probabilities is of integral form which can easily be used for establsing the limit theorem.

The asymptotic properties of the final distribution (1) are considered as $\alpha_3 \to \infty$, because we are interested in the case where, for $t = 0$, the number of infected individuals is small and that of noninfected ones is large. The limit of Theorem 3 belongs to theorems of "threshold" type (see [4], [13]), which are applied to determine the threshold number of infected individuals; exceeding this number means the beginning of an epidemic.

3. Integral representation of the solution of the Kolmogorov system of equations. We introduce the exponential (double) generation function ($|s_1| \leqslant$

$1, |s_2| \leqslant 1, |s_3| \leqslant 1)$

$$\mathcal{F}(t; z_1, z_2, z_3; s_1, s_2, s_3) = \sum_{\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2\beta_3=0}^{\infty} \frac{z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3}}{\alpha_1!\alpha_2!\alpha_3!} P_{(\beta_1,\beta_2,\beta_3)}^{(\alpha_1,\alpha_2,\alpha_3)}(t) s_1^{\beta_1} s_2^{\beta_2} s_3^{\beta_3}.$$

For the considered process we can write the first (direct) and the second (inverse) systems of Kolmogorov differential equations for the transition probabilities $P_{(\beta_1,\beta_2,\beta_3)}^{(\alpha_1,\alpha_2,\alpha_3)}(t)$ in the form of two equations in partial derivatives of second order (see [8], [10])

$$\begin{aligned} \frac{\partial \mathcal{F}}{\partial t} &= \mu_1 z_1 z_3 \left(\frac{\partial \mathcal{F}}{\partial z_1} - \frac{\partial^2 \mathcal{F}}{\partial z_1 \partial z_3} \right) + \mu_2 z_2 z_3 \left(\frac{\partial \mathcal{F}}{\partial z_2} - \frac{\partial^2 \mathcal{F}}{\partial z_2 \partial z_3} \right) \\ &\quad + \rho_1 z_1 \left(\mathcal{F} - \frac{\partial \mathcal{F}}{\partial z_1} \right) + \rho_2 z_2 \left(\mathcal{F} - \frac{\partial \mathcal{F}}{\partial z_2} \right); \end{aligned} \tag{2}$$

$$\begin{aligned} \frac{\partial \mathcal{F}}{\partial t} &= \mu_1 (s_1 - s_1 s_3) \frac{\partial^2 \mathcal{F}}{\partial s_1 \partial s_3} + \mu_2 (s_2 - s_2 s_3) \frac{\partial^2 \mathcal{F}}{\partial s_2 \partial s_3} \\ &\quad + \rho_1 (1 - s_1) \frac{\partial \mathcal{F}}{\partial s_1} + \rho_2 (1 - s_2) \frac{\partial \mathcal{F}}{\partial s_2}, \end{aligned} \tag{3}$$

with the initial condition $\mathcal{F}(0; z_1, z_2, z_3; s_1, s_2, s_3) = e^{z_1 s_1 + z_2 s_2 + z_3 s_3}$.

Using the Fourier variables separation method (see [11], [12]) for the system of linear equations (2), (3) we obtain the solution

$$\begin{aligned} \mathcal{F}(t; z_1, z_2, z_3; s_1, s_2, s_3) = &\sum_{\alpha_1,\alpha_2,\alpha_3=0}^{\infty} \frac{z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3}}{\alpha_1!\alpha_2!\alpha_3!} e^{z_1\rho_1/(\mu_1\alpha_3+\rho_1) + z_2\rho_2/(\mu_2\alpha_3+\rho_2) + z_3} \\ &\times \left(s_1 - \frac{\rho_1}{\mu_1\alpha_3 + \rho_1} \right)^{\alpha_1} \left(s_2 - \frac{\rho_2}{\mu_2\alpha_3 + \rho_2} \right)^{\alpha_2} \\ &\times (s_3 - 1)^{\alpha_3} e^{-(\mu_1\alpha_1\alpha_3 + \mu_2\alpha_2\alpha_3 + \rho_1\alpha_1 + \rho_2\alpha_2)t}. \end{aligned} \tag{4}$$

Absolute convergence of the series (4) for all $z_1, z_2, z_3, s_1, s_2, s_3$ and $t \in [0, \infty)$ is obvious. Furthermore, we need the function ($x > 0$, $y > 0$)

$$H(x, y) = \int_0^\infty \int_0^\infty J_0(2\sqrt{ux}) J_0(2\sqrt{vy})\ {}_0F_2(1, 1; -uv)\, du dv,$$

where $J_0(z) = \sum_{k=0}^{\infty} (-1)^k (z/2)^{2k}/(k!k!)$ is the Bessel function of order zero, and ${}_0F_2(1, 1; z) = \sum_{k=0}^{\infty} z^k/(k!)^3$ is a hypergeometric function.

THEOREM 1. *For the Markov process $\xi(t)$, the double generating function is as follows*:

$$\begin{aligned} \mathcal{F}(t; z_1, z_2, z_3; s_1, s_2, s_3) = &\int_0^\infty \int_0^\infty e^{z_1 s_1 e^{-(\mu_1 x + \rho_1)t} + z_2 s_2 e^{-(\mu_2 x + \rho_2)t} + z_3} \\ \times \Bigg\{ e^{z_3(s_3-1)e^{-y}} &+ \sum_{i=1}^{2} \int_0^\infty e^{-u + z_3(s_3-1)e^{-y - \frac{\mu_i}{\rho_i} u}} \sqrt{\frac{z_i}{u} \left(1 - e^{-(\mu_i x + \rho_i)t}\right)} \\ &\times I_1\left(2\sqrt{z_i u (1 - e^{-(\mu_i x + \rho_i)t})}\right) du \\ + \int_0^\infty \int_0^\infty &e^{-u - v + z_3(s_3-1)e^{-y - \frac{\mu_1}{\rho_1} u - \frac{\mu_2}{\rho_2} v}} \end{aligned}$$

$$\times \sqrt{\frac{z_1 z_2}{uv}\,(1-e^{-(\mu_1 x+\rho_1)t})(1-e^{-(\mu_2 x+\rho_2)t})}$$
$$\times I_1\Big(2\sqrt{z_1 u(1-e^{-(\mu_1 x+\rho_1)t})}\Big)$$
$$\times I_1\Big(2\sqrt{z_2 v(1-e^{-(\mu_2 x+\rho_2)t})}\Big)\,du\,dv\Big\}H(x,y)\,dx\,dy, \tag{5}$$

where $I_1(z)$ is the modified Bessel function.

Proof. To obtain the solution (5) of the system (2), (3) we use the following representation of an exponent (see [14, vol. 2, relation (3.5); vol. 1, Chap. 2, section 12]):

$$e^{-(\alpha_1\mu_1+\alpha_2\mu_2)\alpha_3 t} = \int_0^\infty\int_0^\infty e^{-(\alpha_1\mu_1+\alpha_2\mu_2)tx-\alpha_3 y}\,H(x,y)\,dx\,dy. \tag{6}$$

After substitution of (6) into (4) and changing the order of summation (its validity is implyed by absolute convergence) we get

$$\begin{aligned}
\mathcal{F}(t;z_1,z_2,z_3;s_1,s_2,s_3) &= \sum_{\alpha_1,\alpha_2,\alpha_3=0}^{\infty}\frac{z_1^{\alpha_1}z_2^{\alpha_2}z_3^{\alpha_3}}{\alpha_1!\alpha_2!\alpha_3!}\,e^{z_1\rho_1/(\mu_1\alpha_3+\rho_1)+z_2\rho_2/(\mu_2\alpha_3+\rho_2)+z_3}\\
&\times\left(s_1-\frac{\rho_1}{\mu_1\alpha_3+\rho_1}\right)^{\alpha_1}\left(s_2-\frac{\rho_2}{\mu_2\alpha_3+\rho_2}\right)^{\alpha_2}(s_3-1)^{\alpha_3}\,e^{-(\alpha_1\rho_1+\alpha_2\rho_2)t}\\
&\times\int_0^\infty\int_0^\infty e^{-(\alpha_1\mu_1+\alpha_2\mu_2)tx-\alpha_3 y}H(x,y)\,dx\,dy\\
&=\int_0^\infty\int_0^\infty e^{z_3}\Bigg\{\sum_{\alpha_3=0}^{\infty}\frac{z_3^{\alpha_3}}{\alpha_3!}\,e^{z_1\rho_1/(\alpha_3\mu_1+\rho_1)+z_2\rho_2/(\alpha_3\mu_2+\rho_2)}\\
&\qquad\times(s_3-1)^{\alpha_3}\,e^{-\alpha_3 y}\Bigg\{\sum_{\alpha_1=0}^{\infty}\frac{z_1^{\alpha_1}}{\alpha_1!}\left[\left(s_1-\frac{\rho_1}{\mu_1\alpha_3+\rho_1}\right)e^{-(\mu_1 x+\rho_1)t}\right]^{\alpha_1}\\
&\qquad\times\sum_{\alpha_2=0}^{\infty}\frac{z_2^{\alpha_2}}{\alpha_2!}\left[\left(s_2-\frac{\rho_2}{\mu_2\alpha_3+\rho_2}\right)e^{-(\mu_2 x+\rho_2)t}\right]^{\alpha_2}\Bigg\}\Bigg\}H(x,y)\,dx\,dy\\
&=\int_0^\infty\int_0^\infty e^{z_1 s_1 e^{-(\mu_1 x+\rho_1)t}+z_2 s_2 e^{-(\mu_2 x+\rho_2)t}+z_3}\Bigg\{\sum_{\alpha_3=0}^{\infty}\frac{[z_3(s_3-1)e^{-y}]^{\alpha_3}}{\alpha_3!}\\
&\qquad\times e^{z_1\rho_1(1-e^{-(\mu_1 x+\rho_1)t})/(\alpha_3\mu_1+\rho_1)+z_2\rho_2(1-e^{-(\mu_2 x+\rho_2)t})/(\alpha_3\mu_2+\rho_2)}\Bigg\}\\
&\qquad\times H(x,y)\,dx\,dy.
\end{aligned} \tag{7}$$

For summation of the series in brackets we use the formula

$$\sum_{\alpha=0}^{\infty}\frac{b^\alpha}{\alpha!(\alpha+\mu)^k}=\frac{1}{(k-1)!}\int_0^\infty u^{k-1}e^{-\mu u+be^{-u}}du,\qquad k=1,2,\dots,$$
$$\sum_{\alpha=0}^{\infty}\frac{b^\alpha}{\alpha!(\alpha+\mu)^k(\alpha+\rho)^l}=\frac{1}{(k-1)!(l-1)!}\times\int_0^\infty\int_0^\infty u^{k-1}v^{l-1}e^{-\mu u-\rho v+be^{-u-v}}\,du\,dv,$$

$k, l = 1, 2, \ldots$, and the Bessel function

$$I_1(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{2k+1}}{k!(k+1)!}$$

(substituting $a_1 = z_1\rho_1(1 - e^{-(\mu_1 x+\rho_1)t})/\mu_1$, $a_2 = z_2\rho_2(1 - e^{-(\mu_2 x+\rho_2)t})/\mu_2$, $b = z_3(s_3 - 1)e^{-y}$):

$$\begin{aligned}
&\sum_{\alpha_3=0}^{\infty} \frac{b^{\alpha_3}}{\alpha_3!}\, e^{a_1/(\alpha_3+\rho_1/\mu_1)+a_2/(\alpha_3+\rho_2/\mu_2)} \\
&\quad = \sum_{k,l=0}^{\infty} \frac{a_1^k a_2^l}{k!l!} \sum_{\alpha_3=0}^{\infty} \frac{b^{\alpha_3}}{\alpha_3!(\alpha_3+\rho_1/\mu_1)^k(\alpha_3+\rho_2/\mu_2)^l} \\
&\quad = \sum_{\alpha_3=0}^{\infty} \frac{b^{\alpha_3}}{\alpha_3!} + \sum_{k=1}^{\infty} \frac{a_1^k}{k!} \sum_{\alpha_3=0}^{\infty} \frac{b^{\alpha_3}}{\alpha_3!(\alpha_3+\rho_1/\mu_1)^k} + \sum_{l=1}^{\infty} \frac{a_2^l}{l!} \sum_{\alpha_3=0}^{\infty} \frac{b^{\alpha_3}}{\alpha_3!(\alpha_3+\rho_2/\mu_2)^l} \\
&\qquad + \sum_{k,l=1}^{\infty} \frac{a_1^k a_2^l}{k!l!} \sum_{\alpha_3=0}^{\infty} \frac{b^{\alpha_3}}{\alpha_3!(\alpha_3+\rho_1/\mu_1)^k(\alpha_3+\rho_2/\mu_2)^l} \\
&\quad = e^b + \sum_{i=1}^{2}\sum_{k=1}^{\infty} \frac{a_i^k}{k!(k-1)!} \int_0^{\infty} u^{k-1}\, e^{-\frac{\rho_i}{\mu_i}u+be^{-u}}\, du \\
&\qquad + \sum_{k,l=1}^{\infty} \frac{a_1^k}{k!(k-1)!}\, \frac{a_2^l}{l!(l-1)!} \int_0^{\infty}\int_0^{\infty} u^{k-1}\, v^{l-1}\, e^{-\frac{\rho_1}{\mu_1}u-\frac{\rho_2}{\mu_2}v+be^{-u-v}}\, du\, dv \\
&\quad = e^b + \sum_{i=1}^{2} \int_0^{\infty} \sqrt{\frac{a_i}{u}}\, I_1(2\sqrt{a_i u})\, e^{-\frac{\rho_i}{\mu_i}u+be^{-u}}\, du \\
&\qquad + \int_0^{\infty}\int_0^{\infty} \sqrt{\frac{a_1}{u}}\sqrt{\frac{a_2}{v}}\, I_1(2\sqrt{a_1 u})\, I_1(2\sqrt{a_2 v})\, e^{-\frac{\rho_1}{\mu_1}u-\frac{\rho_2}{\mu_2}v+be^{-u-v}}\, du\, dv. \qquad (8)
\end{aligned}$$

Combining (8) and (7), we obtain (5) (with respect to u, v). This complets the proof of Theorem 1.

By formula (5) we can now obtain expectation and variance for the random processes $\xi_1(t)$, $\xi_2(t)$, $\xi_3(t)$ (see [12]).

4. Final probability distribution and limit theorem. For final probabilities (1) we have generating functions ($|s| \leqslant 1$)

$$\Phi_{(\alpha_1,\alpha_2,\alpha_3)}(s) = \sum_{\gamma_3=0}^{\infty} q_{(0,0,\gamma_3)}^{(\alpha_1,\alpha_2,\alpha_3)} s^{\gamma_3};$$

$$\Phi(z_1, z_2, z_3; s) = \sum_{\alpha_1,\alpha_2,\alpha_3=0}^{\infty} \frac{z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3}}{\alpha_1!\alpha_2!\alpha_3!}\, \Phi_{(\alpha_1,\alpha_2,\alpha_3)}(s), \qquad (9)$$

Passing to the limit $\Phi(z_1, z_2, z_3; s) = \lim_{t\to\infty} \mathcal{F}(t; z_1, z_2, z_3; s_1, s_2, s)$ in (5) leads to the representation

$$\Phi(z_1, z_2, z_3; s) = \int_0^{\infty}\int_0^{\infty} e^{z_3} \Big\{ e^{z_3(s-1)e^{-y}} + \sum_{i=1}^{2} \int_0^{\infty} e^{-u+z_3(s-1)\exp(-y-(\mu_i/\rho_i)u)}$$

$$\times \sqrt{\frac{z_i}{u}}\, I_1(2\sqrt{z_i u})\, du$$

$$+ \int_0^\infty \int_0^\infty e^{-u-v+z_3(s-1)\,\exp(-y-(\mu_1/\rho_1)u-(\mu_2/\rho_2)v)}$$

$$\times \sqrt{\frac{z_1 z_2}{uv}}\, I_1(2\sqrt{z_1 u})\, I_1(2\sqrt{z_2 v})\, du\, dv \Big\} H(x,y)\, dx\, dy.$$

Expanding into power series with respect to z_1, z_2, z_3, taking into account (9), and evaluating the integrals, we see that the following theorem holds true.

THEOREM 2. *For the Markov process $\xi(t)$, the generating function of final probabilities $(\alpha_1 > 0, \alpha_2 > 0)$ is as follows*:

$$\Phi_{(\alpha_1,\alpha_2,\alpha_3)}(s) = \frac{1}{(\alpha_1-1)!(\alpha_2-1)!} \int_0^\infty \int_0^\infty u^{\alpha_1-1}\, v^{\alpha_2-1}$$

$$\times (1 - e^{-(\mu_1/\rho_1)u-(\mu_2/\rho_2)v} + s e^{-(\mu_1/\rho_1)u-(\mu_2/\rho_2)v})^{\alpha_3}\, e^{-u-v}\, du\, dv. \tag{10}$$

Expanding (10) into power series with respect to s, we obtain the following statement.

COROLLARY 1. *Final probabilities for the Becker epidemic process are as follows*:

$$q_{(0,0,\gamma_3)}^{(\alpha_1,\alpha_2,\alpha_3)} = C_{\alpha_3}^{\gamma_3} \sum_{i=0}^{\alpha_3-\gamma_3} (-1)^i C_{\alpha_3-\gamma_3}^i \Big(\frac{\rho_1}{\mu_1 i + \mu_1\gamma_3 + \rho_1}\Big)^{\alpha_1} \Big(\frac{\rho_2}{\mu_2 i + \mu_2\gamma_3 + \rho_2}\Big)^{\alpha_2}.$$

In the particular case of $\rho_1 = \rho_2$ the formula for transition and final probabilities was obtained in [3] by other methods.

In the special case under consideration, particles of type T_3 are said to be *final* [7]. Denote by $\eta^{(\alpha_1,\alpha_2,\alpha_3)}$ a random number of particles of type T_3 which remains after termination of the epidemic process, i.e., there are no longer any particles of types T_1 and T_2. The random variable $\eta^{(\alpha_1,\alpha_2,\alpha_3)}$ has the probability distribution $\{q_{(0,0,\gamma_3)}^{(\alpha_1,\alpha_2,\alpha_3)},\ \gamma_3 = 0,\dots,\alpha_3\}$, which is defined by the generating function (10). For the expectation we obtain

$$\mathbf{E}\,\eta^{(\alpha_1,\alpha_2,\alpha_3)} = \Phi'_{(\alpha_1,\alpha_2,\alpha_3)}(1) = \alpha_3 \Big(\frac{\rho_1}{\mu_1+\rho_1}\Big)^{\alpha_1} \Big(\frac{\rho_2}{\mu_2+\rho_2}\Big)^{\alpha_2}.$$

The calculation of the variance

$$\mathbf{D}\eta^{(\alpha_1,\alpha_2,\alpha_3)} = \Phi''_{(\alpha_1,\alpha_2,\alpha_3)}(1) + \Phi'_{(\alpha_1,\alpha_2,\alpha_3)}(1) - (\Phi'_{(\alpha_1,\alpha_2,\alpha_3)}(1))^2$$

leads to the asymptotic formula, as $\alpha_3 \to \infty$,

$$\mathbf{D}\eta^{(\alpha_1,\alpha_2,\alpha_3)} \sim \alpha_3^2 \Big(\Big(\frac{\rho_1}{2\mu_1+\rho_1}\Big)^{\alpha_1} \Big(\frac{\rho_2}{2\mu_2+\rho_2}\Big)^{\alpha_2} - \Big(\frac{\rho_1}{\mu_1+\rho_1}\Big)^{2\alpha_1} \Big(\frac{\rho_2}{\mu_2+\rho_2}\Big)^{2\alpha_2} \Big).$$

Using the explicit expression (10) for the probability distribution on N, and applying the characteristic functions method in the standard way [7], we obtain the following statement.

THEOREM 3. *Assume $x \in [0,1]$. Then $(\alpha_1 > 0, \alpha_2 > 0)$*

$$F_{(\alpha_1,\alpha_2)}(x) = \lim_{\alpha_3\to\infty} \mathbf{P}\Big\{\frac{\eta^{(\alpha_1,\alpha_2,\alpha_3)}}{\alpha_3} \leqslant x\Big\} = 1 - \frac{1}{(\alpha_1-1)!(\alpha_2-1)!}$$

$$\times \int_0^{-(\rho_1/\mu_1)\ln x} \int_0^{-(\rho_2/\mu_2)\ln x-(\mu_1\rho_2)/(\rho_1\mu_2)u} u^{\alpha_1-1}\, v^{\alpha_2-1}\, e^{-u-v}\, du\, dv.$$

In particular, for $\mu_1\rho_2 \neq \mu_2\rho_1$ *or* $\mu_1\rho_2 = \mu_2\rho_1$, *we have*

$$F_{(1,1)}(x) = \frac{\mu_1\rho_2 x^{\rho_1/\mu_1} - \mu_2\rho_1 x^{\rho_2/\mu_2}}{\mu_1\rho_2 - \mu_2\rho_1} \quad \text{or} \quad F_{(1,1)}(x) = x^{\rho_1/\mu_1}\left(1 - \frac{\rho_1}{\mu_1}\log x\right).$$

accordingly.

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