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ON THE DISTRIBUTION OF THE NUMBER OF FINAL PARTICLES IN A BRANCHING PROCESS WITH TRANSFORMATIONS AND PAIRWISE INTERACTIONS*

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Abstract. A Markov continuous time branching process with two types of particles T_1 and T_2 is considered. Particles of the two types appear either as the offspring of a particle of type T_1, or as a result of interaction of two particles of type T_1. Under certain restrictions on the distribution of the number of new particles the asymptotic behavior of the expectation and variance of the number of particles of the two types are investigated and the asymptotic normality of the distribution of the number of final particles of type T_2 is established when the initial number of particles of type T_1 is large.

Key words. branching process with interaction, final probabilities, exponential generating function, stationary first Kolmogorov equation, explicit solutions

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1. Branching process with two types of particles T_1, T_2 and two kinds of interaction $\varepsilon^1 = (1,0)$, $\varepsilon^2 = (2,0)$ (see [5]). Let $\xi(t)$, $t \in [0,\infty)$, be a time-homogeneous Markov process with state space $\mathbf{N}^2 = \{(\alpha_1,\alpha_2),\ \alpha_1,\alpha_2 = 0,1,2,\ldots\}$. Assume that the transition probabilities $P^{(\alpha_1,\alpha_2)}_{(\beta_1,\beta_2)}(t) = \mathbf{P}(\xi(t) = (\beta_1,\beta_2)\,|\,\xi(0) = (\alpha_1,\alpha_2))$ have, as $\Delta t \to 0$, the form ($\lambda_1 > 0$, $\lambda_2 > 0$):

$$P^{(\alpha_1,\alpha_2)}_{(\alpha_1,\alpha_2)}(\Delta t) = 1 - (\lambda_2\alpha_1(\alpha_1-1) + \lambda_1\alpha_1)\,\Delta t + o(\Delta t),$$
$$P^{(\alpha_1,\alpha_2)}_{(\beta_1,\beta_2)}(\Delta t) = (\lambda_2\alpha_1(\alpha_1-1)\,p^2_{\beta_1-\alpha_1+2,\beta_2-\alpha_2} + \lambda_1\alpha_1 p^1_{\beta_1-\alpha_1+1,\beta_2-\alpha_2})\,\Delta t + o(\Delta t)$$

if $\alpha_1 \neq \beta_1$ or $\alpha_2 \neq \beta_2$. Here $\{p^l_{\gamma_1\gamma_2} \geqq 0,\ \gamma_1,\gamma_2 \in \mathbf{N};\ \sum_{\gamma_1\gamma_2=0}^{\infty} p^l_{\gamma_1\gamma_2} = 1,\ p^l_{l0} = 0\}$, $l = 1,2$, are given probability distributions. Introduce generating functions

$$G_{(\beta_1,\beta_2)}(t;z_1,z_2) = \sum_{\alpha_1,\alpha_2=0}^{\infty} \frac{z_1^{\alpha_1} z_2^{\alpha_2}}{\alpha_1!\,\alpha_2!} P^{(\alpha_1,\alpha_2)}_{(\beta_1,\beta_2)}(t),$$
$$F_{(\alpha_1,\alpha_2)}(t;s_1,s_2) = \sum_{\beta_1,\beta_2=0}^{\infty} P^{(\alpha_1,\alpha_2)}_{(\beta_1,\beta_2)}(t) s_1^{\beta_1} s_2^{\beta_2},$$
$$h_l(s_1,s_2) = \sum_{\gamma_1,\gamma_2=0}^{\infty} p^l_{\gamma_1\gamma_2} s_1^{\gamma_1} s_2^{\gamma_2}, \qquad |s_1| \leqq 1, \quad |s_2| \leqq 1,$$

and linear differential operators

$$h_l\left(\frac{\partial}{\partial z_1},\ \frac{\partial}{\partial z_2}\right) = \sum_{\gamma_1,\gamma_2=0}^{\infty} p^l_{\gamma_1\gamma_2} \frac{\partial^{\gamma_1+\gamma_2}}{\partial z_1^{\gamma_1}\partial z_2^{\gamma_2}}, \qquad l = 1,2.$$

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The first (backward) Kolmogorov system of differential equations for the transition probabilities of the Markov process $\xi(t)$ is equivalent to the following partial differential equation (see [5, Theorem 2]):

(1)
$$\frac{\partial G_{(\beta_1,\beta_2)}}{\partial t} = \left[\lambda_2 z_1^2\left(h_2\left(\frac{\partial}{\partial z_1},\ \frac{\partial}{\partial z_2}\right) - \frac{\partial^2}{\partial z_1^2}\right) + \lambda_1 z_1\left(h_1\left(\frac{\partial}{\partial z_1},\ \frac{\partial}{\partial z_2}\right) - \frac{\partial}{\partial z_1}\right)\right] G_{(\beta_1,\beta_2)}$$

with the initial condition $G_{(\beta_1,\beta_2)}(0; z_1, z_2) = z_1^{\beta_1} z_2^{\beta_2}/(\beta_1!\,\beta_2!)$.

The second (forward) Kolmogorov system of differential equations is equivalent to the following partial differential equation (see [5, Theorem 1]):

$$\frac{\partial F_{(\alpha_1,\alpha_2)}}{\partial t} = \lambda_2(h_2(s_1,s_2) - s_1^2)\,\frac{\partial^2 F_{(\alpha_1,\alpha_2)}}{\partial s_1^2} + \lambda_1(h_1(s_1,s_2) - s_1)\,\frac{\partial F_{(\alpha_1,\alpha_2)}}{\partial s_1}$$

with the initial condition $F_{(\alpha_1,\alpha_2)}(0; s_1, s_2) = s_1^{\alpha_1} s_2^{\alpha_2}$.

A state (α_1, α_2) is treated as existence of α_1 particles of type T_1 and α_2 particles of type T_2 (see [1], [2], [5]). A particle of type T_1 has a random life-length $\tau^1_{(\alpha_1,\alpha_2)}$ with $\mathbf{P}\{\tau^1_{(\alpha_1,\alpha_2)} < t\} = 1 - \exp\{-\lambda_1\alpha_1 t\}$, and, dying, produces, independently of other particles, γ_1 particles of type T_1 and γ_2 particles of type T_2 with probability distribution $\{p^1_{\gamma_1\gamma_2}\}$. Besides, any pair $T_1 + T_1$ of particles transforms after a random time $\tau^2_{(\alpha_1,\alpha_2)}$ with $\mathbf{P}\{\tau^2_{(\alpha_1,\alpha_2)} < t\} = 1 - \exp\{-\lambda_2\alpha_1(\alpha_1 - 1)\,t\}$, independently of other particles, into a group of particles with probability distribution $\{p^2_{\gamma_1\gamma_2}\}$. It is assumed that $\tau^1_{(\alpha_1,\alpha_2)}$ and $\tau^2_{(\alpha_1,\alpha_2)}$ are independent random variables. The process spends in state (α_1, α_2) a random time $\tau_{(\alpha_1,\alpha_2)} = \min(\tau^1_{(\alpha_1,\alpha_2)}, \tau^2_{(\alpha_1,\alpha_2)})$. Particles of type T_2 are called final. Their amount does not decrease in time and has no influence on the number of particles of type T_1 (see [4]).

2. The problem of finding the final probabilities. The states $(0, \beta_2)$, $\beta_2 \in \mathbf{N}$, are absorbing for the Markov process $\xi(t)$. Let $\xi(0) = (\alpha_1, 0)$, $\alpha_1 \in \mathbf{N}$, be the initial state. Introduce the final probabilities $q^{(\alpha_1,0)}_{(0,\beta_2)} = \lim_{t\to\infty} P^{(\alpha_1,0)}_{(0,\beta_2)}(t)$ and the probability of extinction $q_{\alpha_1 0} = \sum_{\beta_2=0}^{\infty} q^{(\alpha_1,0)}_{(0,\beta_2)}$. Denote by η_{α_1} the random number of particles of type T_2 remaining in the process when its evolution stops. The final distribution $\{q^{(\alpha_1,0)}_{(0,\beta_2)}, \beta_2 \in \mathbf{N}\}$ may be improper, i.e., $q_{\alpha_1 0} \leqq 1$.

Introduce the generating functions

$$f_{\alpha_1}(u) = \sum_{\beta_2=0}^{\infty} q^{(\alpha_1,0)}_{(0,\beta_2)}\, u^{\beta_2}, \quad w_0(z,u) = \sum_{\alpha_1=0}^{\infty} \frac{z^{\alpha_1}}{\alpha_1!}\, f_{\alpha_1}(u), \qquad |u| \leqq 1. \tag{2}$$

Set $h^l_{\gamma_1}(u) = \sum_{\gamma_2=0}^{\infty} p^l_{\gamma_1\gamma_2} u^{\gamma_2}$, $|u| \leqq 1$ and let $p^l_{\gamma_1} = \sum_{\gamma_2=0}^{\infty} p^l_{\gamma_1\gamma_2}$, $l = 1, 2$. The criticality parameters of the branching process are denoted by

$$a_l = \lambda_l\left(\frac{\partial h_l(s,u)}{\partial s}\Big|_{s=1,u=1} - l\right), \qquad l = 1, 2$$

(see [4]).

It is known [4, Chap. 2, section 1] that if $\lambda_1 > 0$, $\lambda_2 = 0$ (a branching process with no interaction), then $q_{\alpha_1 0} = q_1^{\alpha_1}$, where q_1 is the closest to a zero nonnegative root of the equation $h_1(s, 1) - s = 0$. The final distribution for subcritical ($a_1 < 0$) and supercritical ($a_1 > 0$) processes is asymptotically normal as $\alpha_1 \to \infty$, while for

the critical processes ($a_1 = 0$) the final distribution is other than normal as $\alpha_1 \to \infty$ [4, Chap. 5, section 5].

In [5] for the case when $\lambda_1 = 0$ and $\lambda_2 > 0$, the asymptotic representation $q_{\alpha_1 0} \sim C_0 q_2^{\alpha_1}$, $\alpha_1 \to \infty$, is found, where $C_0 > 0$ and q_2 is the closest to zero nonnegative root of the equation $h_2(s,1) - s^2 = 0$. In [6] limit theorems are proved for the final distributions in subcritical ($a_2 < 0$), critical ($a_2 = 0$), and supercritical ($a_2 > 0$) processes which are similar to the results established in [4].

A particular case with $\lambda_1 > 0$, $\lambda_2 > 0$ is considered in [7, Corollary 1] where, for a supercritical ($a_2 > 0$) process with $h_1(s,u) = h_0^1(u) + h_2^1(u)\, s^2$, $h_2(s,u) = h_0^2(u) + h_1^2(u)\, s + h_3^2(u)\, s^3$ and under the conditions $p_2^1 > 0$, $p_3^2 > 0$, $p_0^1 + p_0^2 > 0$, $p_0^2 + p_1^2 > 0$, $q_2 > q_1$, the asymptotic representation $q_{\alpha_1 0} \sim C q_2^{\alpha_1}/\alpha_1^{\alpha}$, $\alpha_1 \to \infty$ is found, where $C > 0$ and $\alpha > 0$.

The present paper deals with the subcritical ($a_2 < 0$) process. Explicit expressions for the generating function $f_{\alpha_1}(u)$, asymptotic estimates for the expectation $m_{\alpha_1} = \mathbf{E}\eta_{\alpha_1}$ and the variance $\sigma_{\alpha_1}^2 = \mathbf{D}\eta_{\alpha_1}$ as $\alpha_1 \to \infty$, and asymptotic normality of the final distribution are established under additional constraints on the probability distributions $\{p_{\gamma_1\gamma_2}^l\}$, $l = 1, 2$. The generating function $w_0(z,u)$ is found as an explicit solution of a stationary Kolmogorov equation.

Limit distributions for branching processes with pairwise interaction were considered in [11] and [12].

3. First Kolmogorov equation for the exponential generating function of final probabilities. Introduce the generating function [6]

$$W_{\beta_1}(t; z, u) = \sum_{\alpha_1=0, \beta_2=0}^{\infty} \frac{z^{\alpha_1}}{\alpha_1!}\, P_{(\beta_1,\beta_2)}^{(\alpha_1,0)}(t)\, u^{\beta_2}, \qquad \beta_1 \in \mathbf{N},$$

and the differential operators

$$h_l\left(\frac{\partial}{\partial z_1}, u\right) = \sum_{\gamma_1,\gamma_2=0}^{\infty} p_{\gamma_1\gamma_2}^l u^{\gamma_2} \frac{\partial^{\gamma_1}}{\partial z_1^{\gamma_1}}, \qquad |u| \leqq 1, \quad l = 1, 2.$$

THEOREM 1. *For any $\beta_1 \in \mathbf{N}$, the generating function $W_{\beta_1}(t; z, u)$ of transition probabilities satisfies the following linear partial differential equation*:

$$\frac{\partial W_{\beta_1}}{\partial t} = \left[\lambda_2 z^2 \left(h_2\left(\frac{\partial}{\partial z},\ u\right) - \frac{\partial^2}{\partial z^2}\right) + \lambda_1 z \left(h_1\left(\frac{\partial}{\partial z},\ u\right) - \frac{\partial}{\partial z}\right)\right] W_{\beta_1} \tag{3}$$

with the initial condition $W_{\beta_1}(0; z, u) = z^{\beta_1}/\beta_1!$.

The derivation of (3) is based on (1) and the following relationships for the transition probabilities: $P_{(\beta_1,\beta_2)}^{(\alpha_1,\alpha_2)}(t) = P_{(\beta_1,\beta_2-\alpha_2)}^{(\alpha_1,0)}(t)$ if $\alpha_2 \leqq \beta_2$, $P_{(\beta_1,\beta_2)}^{(\alpha_1,\alpha_2)}(t) = 0$ if $\alpha_2 > \beta_2$ (cf. [6, Theorem 1]).

Similarly to [6] one can show that $w_0(z,u) = \lim_{t\to\infty} W_0(t; z, u)$ and that $w_0(z,u)$ meets the first stationary equation

$$\left[\lambda_2 z \left(h_2\left(\frac{d}{dz},\ u\right) - \frac{d^2}{dz^2}\right) + \lambda_1 \left(h_1\left(\frac{d}{dz},\ u\right) - \frac{d}{dz}\right)\right] w_0(z,u) = 0. \tag{4}$$

The equality $q_{(0,0)}^{(0,0)} = 1$ implies $w_0(0,u) \equiv 1$. Observe that $w_0(z,u)$ is, for $|u| < 1$, an analytical function in variables z and u, since

$$|w_0(z,u)| \leqq \sum_{\alpha_1=0}^{\infty} \frac{|z|^{\alpha_1}}{\alpha_1!} \sum_{\beta_2=0}^{\infty} q_{(0,\beta_2)}^{(\alpha_1,0)}\, |u|^{\beta_2} \leqq \sum_{\alpha_1=0}^{\infty} \frac{|z|^{\alpha_1}}{\alpha_1!} = e^{|z|}.$$

The ordinary differential (4), whose coefficients are linear functions of z, is a Laplace-type equation [10, Vol. 1, Chap. 5, section 22.4]. Its solution is constructed by the method of definite integral

$$w_0(z,u) = \int_K e^{zs}\varphi(s,u)\,ds,$$

where

$$\varphi(s,u) = \frac{1}{h_2(s,u)-s^2}\,\exp\left(\frac{\lambda_1}{\lambda_2}\int\frac{h_1(v,u)-v}{h_2(v,u)-v^2}\,dv\right), \tag{5}$$

and the curve K in the complex s-plane satisfies the condition

$$\int_K \frac{d}{ds}\,e^{zs}(h_2(s,u)-s^2)\,\varphi(s,u)\,ds = 0. \tag{6}$$

To find the form of K it is necessary to consider the Riemann surface of the corresponding function $\varphi(s,u)$ (see [10]).

4. Integral representation for generating function $f_{\alpha_1}(u)$. The case $h_1(s,u) = h_0^1(u) + h_1^1(u)\,s + h_2^1(u)\,s^2$, $h_2(s,u) = h_0^2(u) + h_1^2(u)\,s + h_2^2(u)\,s^2$, $p_0^1 > 0$, $p_2^2 < 1$. Equation (4) for the function $w_0(z,u)$ is an equation of the second order

$$\begin{aligned}\big(\lambda_2(h_2^2(u)-1)\,z + \lambda_1 h_2^1(u)\big)\,\frac{d^2w_0(z,u)}{dz^2} + (\lambda_2 h_1^2(u)\,z + \lambda_1(h_1^1(u)-1))\,\frac{dw_0(z,u)}{dz}\\ + (\lambda_2 h_0^2(u)\,z + \lambda_1 h_0^1(u))\,w_0(z,u) = 0.\end{aligned} \tag{7}$$

Let $h_{\gamma_1}^l(u)$, $\gamma_1 = 0,1,2$, $l = 1,2$, be analytical functions for all u. Denote by $\varphi_0(u)$ and $\varphi_1(u)$ the roots of the quadratic equation $h_2(s,u) - s^2 = 0$. The functions $\varphi_0(u)$ and $\varphi_1(u)$ are analytical in the domain $|u| < 1$ except for a final number of points. Integrating in (5) gives a general solution of (7) in the form ($\varphi_0(u) \neq \varphi_1(u)$),

$$w_0(z,u) = \sum_{n=1}^{2} C_n(u)\int_{K^n} e^{(z+d_0(u))\,s}(s-\varphi_0(u))^{\nu_0(u)-1}(s-\varphi_1(u))^{\nu_1(u)-1}\,ds, \tag{8}$$

where

$$d_0(u) = \frac{\lambda_1}{\lambda_2}\,\frac{h_2^1(u)}{h_2^2(u)-1}, \quad \nu_l(u) = \frac{\lambda_1}{\lambda_2}\,\frac{h_1(\varphi_l(u),u)-\varphi_l(u)}{h_2'(\varphi_l(u),u)-2\varphi_l(u)}, \qquad l = 0,1,$$

and $C_1(u)$ and $C_2(u)$ are arbitrary functions (here we have used the notation $h_2'(s,u) = \partial h_2(s,u)/\partial s$). The curves K^1 and K^2 satisfy condition (6); the integrals over these curves are linearly independent functions in variable z. Note that the functions $\nu_0(u)$ and $\nu_1(u)$, as well as the parameters α and β, are defined (see [7]).

The function $d_0(u)$ is analytical for $|u| < 1$. The analyticity of $\nu_0(u)$, $\nu_1(u)$ in the domain $|u| < 1$ (except for a finite set of points) follows from the equality $h_2'(\varphi_l(u),u) - 2\varphi_l(u) = (-1)^l(h_2^2(u)-1)(\varphi_0(u)-\varphi_1(u))$, $l = 0,1$.

The choice of the integration curves in (8) is related to consideration of the branching points $\varphi_0(u)$ and $\varphi_1(u)$ of the integrands. If K^1 is a curve having the form of a doubled closed contour bypassing the points $\varphi_0(u)$ and $\varphi_1(u)$ first in the counterclockwise and then in the clockwise direction (compare with [9, Chap. 6, section 11,

integral representation (1)]), then integral (8) gives a function analytical in the domain $|u| < 1$, except for a finite set of points at which $\varphi_0(u) = \varphi_1(u)$. The values of $w_0(z,u)$ at the mentioned points are specified by continuity.

Further, it is necessary to use another integral representation for $w_0(z,u)$ being valid in a vicinity of $u = 1$. Since $h_2(1,1) = 1$, we set $\varphi_0(1) = 1$. The conditions $p_2^2 < 1$ and $\varphi_0(1) = 1$ imply $-1 \leqq \varphi_1(1) \leqq 0$. Hence, on account of $h_1(1,1) = 1$ and $p_0^1 > 0$, we obtain $\nu_0(1) = 0$, $\nu_1(1) > 0$. Therefore, in view of the continuity of $\nu_1(u)$ at point $u = 1$, the inequality $\operatorname{Re}\nu_1(u) > 0$ is valid in the domain $|u-1| < \varepsilon$ for an $\varepsilon > 0$. For $\operatorname{Re}\nu_1(u) > 0$ one can take for K^1 a closed contour which starts and ends at point $\varphi_1(u)$ and bypasses point $\varphi_0(u)$ in the counterclockwise direction (compare with [9, Chap. 6, section 11, integral representation (3)]). The curve K^2 is taken as an infinite loop passing along the cut line specified by the ray $\arg(s - \varphi_1(u)) = \theta$, $\operatorname{Re}((z+d_0(u))\,e^{i\theta}) < 0$ and bypassing $\varphi_1(u)$ in the counterclockwise direction (compare with [9, Chap. 6, section 11, integral representation (9)]). Since the function specified by the integral along K^2 is not analytical for $z = -d_0(u)$, we set $C_2(u) \equiv 0$. The function $C_1(u)$ is specified by the condition $w_0(0,u) \equiv 1$. The change of variable $s = \varphi_0(u) + (\varphi_1(u) - \varphi_0(u))\,v$ in (8) gives a representation of the form (in what follows, $d(u) = d_0(u)(\varphi_1(u) - \varphi_0(u))$)

$$w_0(z,u) = \frac{1}{A(u)} \int_1^{(0+)} e^{z(\varphi_0(u)+(\varphi_1(u)-\varphi_0(u))\,v)+d(u)\,v}\, v^{\nu_0(u)-1}(1-v)^{\nu_1(u)-1}\,dv, \tag{9}$$

where

$$A(u) = \int_1^{(0+)} e^{d(u)\,v} v^{\nu_0(u)-1}(1-v)^{\nu_1(u)-1}\,dv, \qquad |u-1| < \varepsilon. \tag{10}$$

THEOREM 2. *Consider a branching process with* $h_1(s,u) = h_0^1(u) + h_1^1(u)\,s + h_2^1(u)\,s^2$, $h_2(s,u) = h_0^2(u) + h_1^2(u)\,s + h_2^2(u)\,s^2$ *and* $p_0^1 > 0$, $p_2^2 < 1$. *The generating function of the final distribution admits for* $|u-1| < \varepsilon$ *the representation*

$$f_{\alpha_1}(u) = \frac{1}{A(u)} \int_1^{(0+)} e^{d(u)\,v}(1-v)^{\nu_1(u)-1}\, \frac{(\varphi_0(u) + (\varphi_1(u) - \varphi_0(u))\,v)^{\alpha_1}}{v^{1-\nu_0(u)}}\,dv. \tag{11}$$

Formula (11) follows from the integral representation (9) and definition (2).

Further, we assume that the integration contour in (10) and (11) consists of an interval directed from $v = 1$ to $v = r$, $0 < r < 1$, the circle $|v| = r$, and the interval directed from $v = r$ to $v = 1$. Since for $u = 1$ the integrands are single-valued functions in the domain $|v| \leqq 1$, the integration over the interval $[r,1]$ in both directions gives zero. Calculating the integrals over the circle $|v| = r$ by the residuals at point $v = 0$, which is a first order pole, we find $A(1) = 2\pi i$ and $f_{\alpha_1}(1) = 1$. In particular, the probability of extinction is $q_{\alpha_1 0} = 1$, $\alpha_1 \in \mathbf{N}$.

Remark 1. Equation (7) reduces to a degenerate hypergeometric equation (see [10, Vol. 3, Chap. 2, Equations (2.113), (2.145)], [8, Chap. 16, Example 5]). The exponential generating function is represented as follows:

$$w_0(z,u) = e^{z\varphi_0(u)}\, \frac{\Phi(\nu_0(u), \nu_0(u) + \nu_1(u);\, z(\varphi_1(u) - \varphi_0(u)) + d(u))}{\Phi(\nu_0(u), \nu_0(u) + \nu_1(u); d(u))}, \tag{12}$$

where

$$\Phi(a,b;z) = 1 + \sum_{n=1}^{\infty} \frac{a(a+1)\cdots(a+n-1)}{b(b+1)\cdots(b+n-1)}\,\frac{z^n}{n!}, \qquad b \neq 0, -1, -2, \ldots, \tag{13}$$

is the degenerate hypergeometric function [9]. For $|u| \leqq 1$, representation (12) holds for all but a finite number of points.

5. Expectation of the final distribution. The case $h_1(s,u) = h_0^1(u) + h_1^1(u)\,s + h_2^1(u)\,s^2$, $h_2(s,u) = h_0^2(u) + h_1^2(u)\,s + h_2^2(u)\,s^2$, $p_0^1 > 0$, $p_2^2 < 1$. We calculate the expectation $m_{\alpha_1} = f'_{\alpha_1}(1)$. Differentiating the ratio (11), we obtain ($\alpha_1 > 0$)

$$m_{\alpha_1} = \frac{1}{2\pi i}\int_1^{(0+)} \frac{d}{du}\left(e^{d(u)\,v}(1-v)^{\nu_1(u)-1}\frac{(\varphi_0(u)+(\varphi_1(u)-\varphi_0(u))\,v)^{\alpha_1}-1}{v^{1-\nu_0(u)}}\right)\Bigg|_{u=1} dv$$

$$(14) = \frac{d'(1)}{2\pi i}\int_1^{(0+)} e^{d(1)\,v}(1-v)^{\nu_1(1)-1}\left((1+(\varphi_1(1)-1)\,v)^{\alpha_1}-1\right) dv$$

$$(15) \quad -\frac{\nu_1'(1)}{2\pi i}\int_1^{(0+)} e^{d(1)\,v}(1-v)^{\nu_1(1)-1}\log(1-v)\cdot\frac{(1+(\varphi_1(1)-1)\,v)^{\alpha_1}-1}{v}\,dv$$

$$+\frac{\nu_0'(1)}{2\pi i}\int_1^{(0+)} e^{d(1)\,v}(1-v)^{\nu_1(1)-1}\log v\cdot\frac{(1+(\varphi_1(1)-1)\,v)^{\alpha_1}-1}{v}\,dv$$

$$+\frac{\alpha_1}{2\pi i}\int_1^{(0+)} e^{d(1)\,v}(1-v)^{\nu_1(1)-1}(\varphi_0'(1)+(\varphi_1'(1)-\varphi_0'(1))\,v)$$

$$(16) \qquad \times\frac{(1+(\varphi_1(1)-1)\,v)^{\alpha_1-1}}{v}\,dv.$$

Integral (14) equals zero since the integrand has no singular points for $|v| < 1$. Calculating the residuals at point $v = 0$, we see that integral (15) equals zero and integral (16) equals $\varphi_0'(1)\,\alpha_1$. Thus, for the expectation we have

(17)

$$m_{\alpha_1} = \varphi_0'(1)\,\alpha_1 - \frac{\nu_0'(1)}{2\pi i}\int_1^{(0+)} e^{d(1)\,v}(1-v)^{\nu_1(1)-1}\log v\,\frac{1-(1-(1-\varphi_1(1))\,v)^{\alpha_1}}{v}\,dv.$$

We transform the integrand in (17) using the sum of the geometric progression $1-(1-(1-\varphi_1(1))\,v)^{\alpha_1} = (1-\varphi_1(1))\,v\sum_{k=0}^{\alpha_1-1}(1-(1-\varphi_1(1))\,v)^k$, and expansions of the polynomial and exponent in terms of powers of $1-v$. We have

$$(1-\varphi_1(1))\int_1^{(0+)} e^{d(1)\,v}(1-v)^{\nu_1(1)-1}\log v\cdot\sum_{k=0}^{\alpha_1-1}((1-\varphi_1(1))(1-v)+\varphi_1(1))^k\,dv$$

$$= e^{d(1)}\sum_{k=0}^{\alpha_1-1}\sum_{l=0}^{k} C_k^l(1-\varphi_1(1))^{l+1}\varphi_1^{k-l}(1)$$

$$(18) \quad \times\sum_{m=0}^{\infty}\frac{(-d(1))^m}{m!}\int_1^{(0+)}(1-v)^{\nu_1(1)+l+m-1}\log v\,dv.$$

We integrate by parts the contour integral and, calculating the residual at point $v = 0$, pass to an integral over the real half-line:

$$\frac{1}{2\pi i}\int_1^{(0+)}(1-v)^{\nu_1(1)+l+m-1}\log v\,dv = \frac{1}{2\pi i}\int_1^{(0+)}\frac{(1-v)^{\nu_1(1)+l+m}}{(\nu_1(1)+l+m)\,v}\,dv$$

$$= \frac{1}{\nu_1(1)+l+m} = \int_0^{+\infty} e^{-(\nu_1(1)+l+m)\,v}\,dv.$$

Substitution of the last integral into (18) and summation of (17) leads to the following expression:

(19)
$$m_{\alpha_1} = \varphi_0'(1)\,\alpha_1 - \nu_0'(1)\int_0^{+\infty} e^{d(1)(1-e^{-v})-\nu_1(1)\,v}\,\frac{1-((1-\varphi_1(1))\,e^{-v}+\varphi_1(1))^{\alpha_1}}{1-e^{-v}}\,dv.$$

THEOREM 3. *Let the conditions of Theorem 2 be valid. Then the expectation of the final distribution has the following asymptotic representation:*

$$m_{\alpha_1} = \varphi_0'(1)\,\alpha_1 - \nu_0'(1)\,\log\alpha_1 + O(1), \qquad \alpha_1\to\infty. \tag{20}$$

Proof. Using the representation

$$\log x = \int_0^{+\infty}\frac{e^{-v}-e^{-xv}}{v}\,dv, \qquad x>0,$$

[9, Chap. 1, section 7, formula (18)], we split the integral in (19) into the sum of integrals. Then

$$\int_0^{+\infty} e^{d(1)(1-e^{-v})-\nu_1(1)\,v}\,\frac{1-((1-\varphi_1(1))\,e^{-v}+\varphi_1(1))^{\alpha_1}}{1-e^{-v}}\,dv$$

$$= \log((1-\varphi_1(1))\,\alpha_1+\nu_1(1)) + \int_0^{+\infty}\left(\frac{e^{d(1)(1-e^{-v})-\nu_1(1)\,v}}{1-e^{-v}} - \frac{e^{-v}}{v}\right)dv \tag{21}$$

$$+ \int_0^{+\infty}\left(\frac{1}{v}-\frac{1}{1-e^{-v}}\right)e^{d(1)(1-e^{-v})-\nu_1(1)\,v}\,((1-\varphi_1(1))\,e^{-v}+\varphi_1(1))^{\alpha_1}\,dv \tag{22}$$

$$+ \int_0^{+\infty}\frac{1-e^{d(1)(1-e^{-v})}}{v}\,e^{-((1-\varphi_1(1))\,\alpha_1+\nu_1(1))\,v}\,dv \tag{23}$$

$$+ \int_0^{\log(1-1/\varphi_1(1))}\frac{e^{-(1-\varphi_1(1))\,\alpha_1 v}-((1-\varphi_1(1))\,e^{-v}+\varphi_1(1))^{\alpha_1}}{v}$$

$$\times\, e^{d(1)(1-e^{-v})-\nu_1(1)\,v}\,dv \tag{24}$$

$$+ \int_{\log(1-1/\varphi_1(1))}^{+\infty}\frac{e^{-(1-\varphi_1(1))\,\alpha_1 v}-((1-\varphi_1(1))\,e^{-v}+\varphi_1(1))^{\alpha_1}}{v}$$

$$\times\, e^{d(1)(1-e^{-v})-\nu_1(1)\,v}\,dv. \tag{25}$$

Summands (24) and (25) vanish if $\varphi_1(1)=0$. The integrands in integrals (21)–(24) are bounded as $v\to 0+$. Hence the integrals are convergent. Integrals (22), (23), and (25) tend to zero as $\alpha_1\to\infty$. To evaluate integral (24) we make use of the inequality $1-x^{\alpha_1}\leqq\alpha_1(1-x)$ for $0<x<1$ and the boundness of the function

$$y(v) = \frac{e^{-(1-\varphi_1(1))\,v}-(1-\varphi_1(1))\,e^{-v}-\varphi_1(1)}{ve^{-(1-\varphi_1(1))\,v}}, \qquad 0<v<\log\left(1-\frac{1}{\varphi_1(1)}\right)$$

(since $\lim_{v\to 0} y(v) = 0$). For (24) we have

$$\int_0^{\log(1-1/\varphi_1(1))} \left(1 - \left(\frac{(1-\varphi_1(1))\, e^{-v} + \varphi_1(1)}{e^{-(1-\varphi_1(1))\, v}}\right)^{\alpha_1}\right) \frac{e^{d(1)(1-e^{-v})-\nu_1(1)\, v-(1-\varphi_1(1))\, \alpha_1 v}}{v}\, dv$$
$$\leqq \alpha_1 \int_0^{\log(1-1/\varphi_1(1))} y(v)\, e^{d(1)(1-e^{-v})-\nu_1(1)\, v-(1-\varphi_1(1))\, \alpha_1 v}\, dv$$
$$\leqq \alpha_1 e^{d(1)} \sup y(v) \int_0^{\log(1-1/\varphi_1(1))} e^{-(1-\varphi_1(1))\, \alpha_1 v}\, dv \leqq \frac{e^{d(1)}}{1-\varphi_1(1)} \sup y(v).$$

The boundness of the values of integrals (21)–(25) and the equality $\log((1-\varphi_1(1))\, \alpha_1 + \nu_1(1)) = \log \alpha_1 + O(1)$, $\alpha_1 \to \infty$, justifies the validity of (20). The theorem is proved.

For the case $h_2(s,u) = p_0^2 + p_1^2 s$ (pairwise interactions give no particles of type T_2) we have $\varphi_0(u) \equiv 1$, $\varphi_1(u) \equiv -p_0^2$, and $m_{\alpha_1} = (\lambda_1/\lambda_2)\, a_2^1 \log \alpha_1/(1+p_0^2) + O(1)$, $\alpha_1 \to \infty$, where $a_2^1 = (\partial h_1(s,u)/\partial u)\,|_{s=1,u=1}$ is the expectation of the number of particles of type T_2, appearing when a particle of type T_1 dies out [12].

It is still unclear whether is it possible or not to find, based on representation (11), an asymptotic representation, as $\alpha_1 \to \infty$ for the variance of the final distribution.

6. Limit theorem for the final distribution. The case $h_1(s,u) = h_0^1(u) + h_1^1(u)\, s$, $h_2(s,u) = h_1^2(u)\, s + h_2^2(u)\, s^2$, $p_0^1 > 0$, $p_2^2 < 1$. In the case under consideration, $\varphi_0(u) = h_1^2(u)/(1-h_2^2(u))$, $\varphi_1(u) \equiv 0$, and $d(u) \equiv 0$. On account of the equality $\Phi(a,b;z) = e^z \Phi(b-a,b;-z)$ [9, Chap. 6, section 3] and (12) we have $w_0(z,u) = \Phi(\nu_1(u), \nu_0(u)+\nu_1(u);\, \varphi_0(u)\, z)$. Recalling (13) and equating coefficients of one and the same powers of z in the series for $w_0(z,u)$ (derived by the above mentioned representation) and the series (2), we obtain the generating function of the final distribution ($\alpha_1 > 0$):

$$f_{\alpha_1}(u) = \varphi_0^{\alpha_1}(u)\, \frac{\nu_1(u)\cdots(\nu_1(u)+\alpha_1-1)}{(\nu_0(u)+\nu_1(u))\cdots(\nu_0(u)+\nu_1(u)+\alpha_1-1)}, \qquad |u| \leqq 1. \tag{26}$$

Taking into account the property of the Gamma-function $\Gamma(z+\alpha_1) = z(z+1)\cdots(z+\alpha_1-1)\,\Gamma(z)$, we deduce the representation

$$f_{\alpha_1}(u) = \varphi_0^{\alpha_1}(u)\, \frac{\Gamma(\nu_1(u)+\alpha_1)\Gamma(\nu_0(u)+\nu_1(u))}{\Gamma(\nu_0(u)+\nu_1(u)+\alpha_1)\Gamma(\nu_1(u))}. \tag{27}$$

Using the equalities $\nu_1(u) = (\lambda_1/\lambda_2) h_0^1(u)/h_1^2(u)$, $\nu_0(u)+\nu_1(u) = (\lambda_1/\lambda_2)(1-h_1^1(u))/(1-h_2^2(u))$, one can establish the analyticity of the function (26) in the domain $|u| < 1+\varepsilon$ for some $\varepsilon > 0$. Let us calculate the expectation $m_{\alpha_1} = f'_{\alpha_1}(1)$ and variance $\sigma^2_{\alpha_1} = f''_{\alpha_1}(1) + f'_{\alpha_1}(1) - (f'_{\alpha_1}(1))^2$. Differentiating (27), we make use of the function $\psi(z) = d\log\Gamma(z)/dz$ [9, Chap. 1, section 7], [8, Chap. 12]. As a result we get

$$m_{\alpha_1} = \varphi_0'(1)\, \alpha_1 - \nu_0'(1)\big(\psi(\nu_1(1)+\alpha_1) - \psi(\nu_1(1))\big), \tag{28}$$

$$\sigma^2_{\alpha_1} = \big(\varphi_0''(1) + \varphi_0'(1) - (\varphi_0'(1))^2\big)\, \alpha_1 - \big(\nu_0''(1)+\nu_0'(1)\big)\big(\psi(\nu_1(1)+\alpha_1) - \psi(\nu_1(1))\big) \tag{29}$$

$$- \nu_0'(1)\big(\nu_0'(1) + 2\nu_1'(1)\big)\big(\psi'(\nu_1(1)+\alpha_1) - \psi'(\nu_1(1))\big). \tag{30}$$

Formula (28) can also be deduced by (19) and the integral representation for $\psi(z)$ in [9, Chap. 1, section 7, formula (17)].

THEOREM 4. *Let a branching process with* $h_1(s,u) = h_0^1(u) + h_1^1(u)\, s$, $h_2(s,u) = h_1^2(u)\, s + h_2^2(u)\, s^2$ *and* $p_0^1 > 0$, $p_2^2 < 1$ *be given. Then the expectation and variance of the final distribution have the asymptotic formulas*

$$m_{\alpha_1} \sim \varphi_0'(1)\, \alpha_1 - \nu_0'(1)\, \log \alpha_1 + \nu_0'(1)\, \psi(\nu_1(1)), \tag{31}$$

$$\sigma_{\alpha_1}^2 \sim (\varphi_0''(1) + \varphi_0'(1) - (\varphi_0'(1))^2)\, \alpha_1 - (\nu_0''(1) + \nu_0'(1))\, \log \alpha_1, \qquad \alpha_1 \to \infty. \tag{32}$$

To derive (31) and (32) it is necessary to use (28) and (30) and the asymptotic formulas $\psi(z) \sim \log z$, $\psi'(z) \sim 1/z$ as $z \to \infty$, $|\arg z| < \pi$ (see [9, Chap. 1, section 18]).

THEOREM 5. *Let the conditions of Theorem 4 be valid, and either* $\partial h_1(s,u)/\partial u \not\equiv 0$ *or* $\partial h_2(s,u)/\partial u \not\equiv 0$. *For any fixed* $x \in (-\infty, \infty)$,

$$\lim_{\alpha_1 \to \infty} \mathbf{P}\left\{ \frac{\eta_{\alpha_1} - m_{\alpha_1}}{\sigma_{\alpha_1}} < x \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy. \tag{33}$$

Proof. We show that the characteristic function $\hat{f}_{\alpha_1}(\tau) = \mathbf{E} \exp(i\tau(\eta_{\alpha_1} - m_{\alpha_1})/\sigma_{\alpha_1})$ of the scaled random variable $(\eta_{\alpha_1} - m_{\alpha_1})/\sigma_{\alpha_1}$ converges, as $\alpha_1 \to \infty$, to $\exp(-\tau^2/2)$, which is the characteristic function of the standard normal distribution [3].

Introduce the function $y_{\alpha_1}(\tau) = \exp(i\tau/\sigma_{\alpha_1})$. Then $\hat{f}_{\alpha_1}(\tau) = y_{\alpha_1}(-m_{\alpha_1}\tau)\, f_{\alpha_1}(y_{\alpha_1}(\tau))$ and (27) yields

$$\hat{f}_{\alpha_1}(\tau) = y_{\alpha_1}(-m_{\alpha_1}\tau)\, \varphi_0^{\alpha_1}(y_{\alpha_1}(\tau))\, \frac{\Gamma(\nu_1(y_{\alpha_1}(\tau)) + \alpha_1)\Gamma(\nu_0(y_{\alpha_1}(\tau)) + \nu_1(y_{\alpha_1}(\tau)))}{\Gamma(\nu_0(y_{\alpha_1}(\tau)) + \nu_1(y_{\alpha_1}(\tau)) + \alpha_1)\Gamma(\nu_1(y_{\alpha_1}(\tau)))}.$$

Using the Taylor expansion $\log \varphi_0(e^v) = \varphi_0'(1)\, v + (\varphi_0''(1) + \varphi_0'(1) - (\varphi_0'(1))^2)\, v^2/2 + o(v^2)$, $v \to 0$, we obtain as $\alpha_1 \to \infty$

$$\varphi_0^{\alpha_1}(y_{\alpha_1}(\tau)) = \exp\left(i\tau\, \frac{\varphi_0'(1)\, \alpha_1}{\sigma_{\alpha_1}} - \tau^2\, \frac{(\varphi_0''(1) + \varphi_0'(1) - (\varphi_0'(1))^2)\, \alpha_1}{2\sigma_{\alpha_1}^2} + o\left(\frac{\alpha_1}{\sigma_{\alpha_1}^2}\right)\right). \tag{34}$$

By Stirling's formula $\Gamma(z) \sim \sqrt{2\pi}\, \exp((z - \frac{1}{2})\, \log z - z)$, $z \to \infty$, $|\arg z| < \pi$, we deduce

$$\begin{aligned} &\frac{\Gamma(\nu_1(y_{\alpha_1}(\tau)) + \alpha_1)}{\Gamma(\nu_0(y_{\alpha_1}(\tau)) + \nu_1(y_{\alpha_1}(\tau)) + \alpha_1)} \\ &\quad \sim \exp\left(\left(\nu_1(y_{\alpha_1}(\tau)) + \alpha_1 - \frac{1}{2} \right) \log \left(1 - \frac{\nu_0(y_{\alpha_1}(\tau))}{\nu_0(y_{\alpha_1}(\tau)) + \nu_1(y_{\alpha_1}(\tau)) + \alpha_1} \right) \right. \\ &\qquad \left. - \nu_0(y_{\alpha_1}(\tau))\, \log(\nu_0(y_{\alpha_1}(\tau)) + \nu_1(y_{\alpha_1}(\tau)) + \alpha_1) + \nu_0(y_{\alpha_1}(\tau)) \right). \end{aligned} \tag{35}$$

Since $\log(1+z) \sim z$, $z \to 0$, and $\nu_0(y_{\alpha_1}(\tau)) \to 0$, $\nu_1(y_{\alpha_1}(\tau)) \to \nu_1(1)$ as $\alpha_1 \to \infty$, the first summand under the exponent sign in (35) vanishes. Since $\log(\nu_0(y_{\alpha_1}(\tau)) + \nu_1(y_{\alpha_1}(\tau)) + \alpha_1) \sim \log \alpha_1$, $\alpha_1 \to \infty$, we have

$$\frac{\Gamma(\nu_1(y_{\alpha_1}(\tau)) + \alpha_1)}{\Gamma(\nu_0(y_{\alpha_1}(\tau)) + \nu_1(y_{\alpha_1}(\tau)) + \alpha_1)} \sim \exp(-\nu_0(y_{\alpha_1}(\tau))\, \log \alpha_1). \tag{36}$$

Using the Taylor expansion $\nu_0(e^v) = \nu_0'(1)\, v + (\nu_0''(1) + \nu_0'(1))\, v^2/2 + o(v^2)$, $v \to 0$,

we derive from (36) as $\alpha_1 \to \infty$

$$\begin{aligned}&\frac{\Gamma(\nu_1(y_{\alpha_1}(\tau))+\alpha_1)}{\Gamma(\nu_0(y_{\alpha_1}(\tau))+\nu_1(y_{\alpha_1}(\tau))+\alpha_1)}\\ &\quad=\exp\biggl(-i\tau\,\frac{\nu_0'(1)\,\log\alpha_1}{\sigma_{\alpha_1}}+\tau^2\,\frac{(\nu_0''(1)+\nu_0'(1))\,\log\alpha_1}{2\sigma_{\alpha_1}^2}+o\biggl(\frac{\log\alpha_1}{\sigma_{\alpha_1}^2}\biggr)\biggr).\end{aligned}\tag{37}$$

Multiplying $y_{\alpha_1}(-m_{\alpha_1}\tau)$, (34), and (37) and observing that

$$\Gamma(\nu_0(y_{\alpha_1}(\tau))+\nu_1(y_{\alpha_1}(\tau)))\sim\Gamma(\nu_1(y_{\alpha_1}(\tau))),\qquad \alpha_1\to\infty,$$

we have for $h_2(s,u)\neq s$

$$\begin{aligned}\hat f_{\alpha_1}(\tau)=\exp\biggl(&i\tau\,\frac{\varphi_0'(1)\,\alpha_1-\nu_0'(1)\,\log\alpha_1-m_{\alpha_1}}{\sigma_{\alpha_1}}\\ &-\tau^2\,\frac{(\varphi_0''(1)+\varphi_0'(1)-(\varphi_0'(1))^2)\,\alpha_1-(\nu_0''(1)+\nu_0'(1))\,\log\alpha_1}{2\sigma_{\alpha_1}^2}+o\biggl(\frac{\alpha_1}{\sigma_{\alpha_1}^2}\biggr)\biggr).\end{aligned}\tag{38}$$

If $h_2(s,u)=s$, we see that $\varphi_0(u)\equiv 1$, and the remainder term in the last formula is $o(\log\alpha_1/\sigma_{\alpha_1}^2)$. Estimate (31) implies that the first summand under the exponent sign in (38) tends to zero. Applying to the second summand and the remainder term the asymptotic representation (32), we obtain

$$\lim_{\alpha_1\to\infty}\hat f_{\alpha_1}(\tau)=\exp\biggl(-\frac{\tau^2}{2}\biggr).$$

The theorem is proved.

Remark 2. The probability generating function (26) can be written as a product

$$f_{\alpha_1}(u)=g_1(u)\cdots g_{\alpha_1}(u),$$

where $g_n(u)=\varphi_0(u)(\nu_1(u)+n-1)/(\nu_0(u)+\nu_1(u)+n-1)$, $n=1,\dots,\alpha_1$, are probability generating functions. Applying Lyapunov's theorem [3, section 40] to the sum $\eta_{\alpha_1}=\eta^1+\cdots+\eta^{\alpha_1}$ of independent random variables η^n with generating functions $g_n(u)$ aiming to derive (33) requires us to verify the validity of the conditions of the mentioned theorem.

REFERENCES

[1] A. T. Bharucha-Reid, *Elements of the Theory of Markov Processes and Their Applications*, McGraw-Hill, New York, Toronto, London, 1960.

[2] N. G. Van Kampen, *Stochastic Processes in Physics and Chemistry*, Lecture Notes in Mathematics 888, North-Holland, Amsterdam, New York, 1981.

[3] B. V. Gnedenko, *The Theory of Probability*, Chelsea, New York, 1967.

[4] B. A. Sevastyanov, *Verzweigungsprozesse*, Springer-Verlag, Berlin, Heidelberg, New York, 1975.

[5] B. A. Sevastyanov and A. V. Kalinkin, *Branching processes with interaction of particles*, Dokl. Akad. Nauk SSSR, 264 (1982), pp. 306–308 (in Russian).

[6] A. V. Kalinkin, *Final probabilities for a branching random process with interaction of particles*, Dokl. Akad. Nauk SSSR, 269 (1983), pp. 1309–1312.

[7] A. V. Kalinkin *On the extinction probability of a branching processes with two kinds of interaction of particles*, Theory Probab. Appl., 46 (2002), pp. 347–352.

[8] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, *Vol.* 2: *Principal Trancendental Functions*, Cambridge University Press, Cambridge, UK, 1952.

[9] G. Bateman and A. Erdélyi, *Higher Transcendental Functions*, *Vol.* 1: *The Hypergeometric Function. Legendre Functions*, McGraw-Hill, New York, 1953.

[10] E. Kamke, *Differentialgleichungen*, *Lösungsmethoden*, *Vol.* I, Chelsea, New York, 1974.

[11] A. M. Lange, *Stationary distribution in an open stochastic system with pairwise interaction of particles*, Vestnik MGTU Ser. Estestv Nauk, 16 (2005), pp. 3–22 (in Russian).

[12] A. M. Lange, *Asymptotic properties of final probabilities of a branching process with interaction of particles*, Obozr. Prikl. Prom. Mat., 12 (2005), pp. 669–770 (in Russian).