

Final Distribution for Gani Epidemic Markov Processes

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Abstract—We consider the Kolmogorov equations for the transition probabilities of a three-dimensional Markov process of special form. For a stationary first equation, an exact solution is obtained using the Riemann method. We obtain asymptotics for the expectation and variance of the final distribution and establish a limit theorem.

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1. EPIDEMIC PROCESS

On the set of states

$$\mathbb{N}^3 = \{\alpha = (\alpha_1, \alpha_2, \alpha_3), \alpha_1, \alpha_2, \alpha_3 = 0, 1, 2, \dots\},$$

we consider a homogeneous (in time) Markov process

$$\xi(t) = (\xi_1(t), \xi_2(t), \xi_3(t)), \quad t \in [0, \infty),$$

with transition probabilities

$$P_{(\beta_1, \beta_2, \beta_3)}^{(\alpha_1, \alpha_2, \alpha_3)}(t) = \mathbb{P}\{\xi(t) = (\beta_1, \beta_2, \beta_3) \mid \xi(0) = (\alpha_1, \alpha_2, \alpha_3)\}.$$

Suppose that the transition probabilities have the following form as $t \rightarrow 0+$:

$$\begin{aligned} P_{(\alpha_1, \alpha_2-1, \alpha_3+1)}^{(\alpha_1, \alpha_2, \alpha_3)}(t) &= \lambda_1 \alpha_1 \alpha_2 t + o(t), \\ P_{(\alpha_1, \alpha_2, \alpha_3-1)}^{(\alpha_1, \alpha_2, \alpha_3)}(t) &= \lambda_2 \alpha_1 \alpha_3 t + o(t), \\ P_{(\alpha_1-1, \alpha_2, \alpha_3)}^{(\alpha_1, \alpha_2, \alpha_3)}(t) &= \lambda_3 \alpha_1 t + o(t), \\ P_{(\alpha_1, \alpha_2, \alpha_3)}^{(\alpha_1, \alpha_2, \alpha_3)}(t) &= 1 - (\lambda_1 \alpha_1 \alpha_2 + \lambda_2 \alpha_1 \alpha_3 + \lambda_3 \alpha_1)t + o(t), \end{aligned}$$

where $\lambda_1 > 0, \lambda_2 > 0, \lambda_3 > 0$. Let us introduce the generating functions of the transition probabilities:

$$F_\alpha(t; s_1, s_2, s_3) = \sum_{\beta_1, \beta_2, \beta_3=0}^{\infty} P_{(\beta_1, \beta_2, \beta_3)}^{(\alpha_1, \alpha_2, \alpha_3)}(t) s_1^{\beta_1} s_2^{\beta_2} s_3^{\beta_3}, \quad |s_1| \leq 1, \quad |s_2| \leq 1, \quad |s_3| \leq 1.$$

The second (direct) system of Kolmogorov differential equations for the transition probabilities of the process $\xi(t)$ is equivalent to the partial differential equation [1], [2]

$$\frac{\partial F_\alpha}{\partial t} = \lambda_1(s_1 s_3 - s_1 s_2) \frac{\partial^2 F_\alpha}{\partial s_1 \partial s_2} + \lambda_2(s_1 - s_1 s_3) \frac{\partial^2 F_\alpha}{\partial s_1 \partial s_3} + \lambda_3(1 - s_1) \frac{\partial F_\alpha}{\partial s_1} \quad (1)$$

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with initial condition $F_\alpha(0; s_1, s_2, s_3) = s_1^{\alpha_1} s_2^{\alpha_2} s_3^{\alpha_3}$. We introduce the exponential (double) generating function of the transition probabilities

$$\mathcal{F}(t; z_1, z_2, z_3; s_1, s_2, s_3) = \sum_{\alpha_1, \alpha_2, \alpha_3=0}^{\infty} \frac{z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3}}{\alpha_1! \alpha_2! \alpha_3!} F_\alpha(t; s_1, s_2, s_3).$$

The first (inverse) system of Kolmogorov differential equations for the transition probabilities can be reduced to the form [2], [3]

$$\frac{\partial \mathcal{F}}{\partial t} = \lambda_1 z_1 z_2 \left(\frac{\partial^2 \mathcal{F}}{\partial z_1 \partial z_3} - \frac{\partial^2 \mathcal{F}}{\partial z_1 \partial z_2} \right) + \lambda_2 z_1 z_3 \left(\frac{\partial \mathcal{F}}{\partial z_1} - \frac{\partial^2 \mathcal{F}}{\partial z_1 \partial z_3} \right) + \lambda_3 z_1 \left(\mathcal{F} - \frac{\partial \mathcal{F}}{\partial z_1} \right)$$

with initial condition

$$\mathcal{F}(0; z_1, z_2, z_3; s_1, s_2, s_3) = e^{z_1 s_1 + z_2 s_2 + z_3 s_3}.$$

The Markov process $\xi(t)$ can be interpreted [1], [4], [5] as a model of the propagation of infection in a population with three types of individuals (particles) and two stages of disease. Particles of type T_1 are infected individuals (the sources of infection); particles of type T_2 are healthy individuals (susceptible to disease, having had no contact with an infected individual); and particles of type T_3 are individuals having been in a single contact with an infected individual. A healthy individual after two contacts with an infected one is eliminated from the population. The state of the process $(\alpha_1, \alpha_2, \alpha_3)$ means the presence of α_1 particles of type T_1 , α_2 particles of type T_2 , and α_3 particles of type T_3 . In random time τ_α^1 , $P\{\tau_\alpha^1 < t\} = 1 - e^{-\alpha_1 \alpha_2 \lambda_1 t}$, a pair of particles of type T_1 and type T_2 interact and transform into a particle of type T_1 and a particle of type T_3 . The process passes to the state corresponding to the vector $(\alpha_1, \alpha_2 - 1, \alpha_3 + 1)$. In random time τ_α^2 , $P\{\tau_\alpha^2 < t\} = 1 - e^{-\alpha_1 \alpha_3 \lambda_2 t}$, a pair of particles of type T_1 and type T_3 interact and transform into a particle of type T_1 . The process passes to the state corresponding to the vector $(\alpha_1, \alpha_2, \alpha_3 - 1)$. Besides, in random time τ_α^3 , $P\{\tau_\alpha^3 < t\} = 1 - e^{-\alpha_1 \lambda_3 t}$, one particle of type T_1 dies and the process passes to the state corresponding to the vector $(\alpha_1 - 1, \alpha_2, \alpha_3)$. The random variables $\tau_\alpha^1, \tau_\alpha^2, \tau_\alpha^3$ are independent and the process is in the state $(\alpha_1, \alpha_2, \alpha_3)$ during random time $\tau_\alpha = \min\{\tau_\alpha^1, \tau_\alpha^2, \tau_\alpha^3\}$. Further, the process evolves along similar lines.

The transition probabilities

$$\{P_{(\beta_1, \beta_2, \beta_3)}^{(\alpha_1, \alpha_2, \alpha_3)}(t), (\beta_1, \beta_2, \beta_3) \in \mathbb{N}^3\}$$

define an α -particle distribution function [6]. For Markov processes of this type, the first system of differential equations is written as a *chain of equations* [7, Chap. 5].

2. THE PROBLEM OF FINAL PROBABILITIES

The Markov process $\xi(t)$ was introduced by Gani [1] and is a generalization of the Weiss–Markov epidemic process [3], [5]. In [1], the second Kolmogorov equation (1) was solved by the Laplace transform method for the parameter values $\lambda_1 = \lambda_2$. The same method was applied in [2] to solve the equations of the Bartlett–McKendrick Markov epidemic process [4]. However, the expressions for the solutions given in [1], [8], consisting of collections of multiple sums and products, are of little use for the study of asymptotic properties of the random processes under consideration.

For the process $\xi(t)$, define final probabilities for absorbing states $(0, \gamma_2, \gamma_3), \gamma_2, \gamma_3 = 0, 1, 2, \dots$,

$$q_{(0, \gamma_2, \gamma_3)}^{(\alpha_1, \alpha_2, \alpha_3)} = \lim_{t \rightarrow \infty} P_{(0, \gamma_2, \gamma_3)}^{(\alpha_1, \alpha_2, \alpha_3)}(t), \quad (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3, \quad \sum_{\gamma_2, \gamma_3=0}^{\infty} q_{(0, \gamma_2, \gamma_3)}^{(\alpha_1, \alpha_2, \alpha_3)} = 1.$$

The problem of determining the final probability distribution for a Markov process on \mathbb{N}^2 was solved in [9] in the special case of a *branching process* when the transition probabilities are nonlinearly related and the equation for the one-particle generating function of the transition probabilities [10] is known.

In the present paper, the Kolmogorov equations for the process $\xi(t)$ are studied by using the exponential generating function of the transition probabilities introduced in [2], and the calculation

of the final distribution for the Markov process can be reduced to the solution of the stationary first equation. Such a method for determining the final probabilities was applied to epidemic processes [3] and other Markov processes on \mathbb{N} and \mathbb{N}^2 (see Chap. 3 in the survey [7]). Our examples of the solutions of stationary Kolmogorov equations are of integral form (see expression (18) of the present paper) which can easily be used to obtain limit theorems.

The asymptotic properties of the final distribution for the process $\xi(t)$ are considered as $\alpha_2 \rightarrow \infty$, because the case in which, for $t = 0$, the number of infected individuals is small and that of noninfected ones is large [5] is of interest. The limit Theorem 2 belongs to theorems of "threshold" type [5], which are applied to define the threshold number of infected individuals; exceeding this number means the beginning of an epidemic.

Results similar to those given in the present paper were described in [11] for the parameter values $\lambda_1 = \lambda_2$.

3. THE STATIONARY FIRST KOLMOGOROV EQUATION

We introduce the generating function of the final probabilities

$$\Phi_{(\alpha_1, \alpha_2, \alpha_3)}(s_2, s_3) = \sum_{\gamma_2, \gamma_3=0}^{\infty} q_{(0, \gamma_2, \gamma_3)}^{(\alpha_1, \alpha_2, \alpha_3)} s_2^{\gamma_2} s_3^{\gamma_3}, \quad |s_2| \leq 1, \quad |s_3| \leq 1,$$

and the exponential generating function

$$\Phi(z_1, z_2, z_3; s_2, s_3) = \sum_{\alpha_1, \alpha_2, \alpha_3=0}^{\infty} \frac{z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3}}{\alpha_1! \alpha_2! \alpha_3!} \Phi_{(\alpha_1, \alpha_2, \alpha_3)}(s_2, s_3). \quad (2)$$

Obviously, the functions $\Phi_{(\alpha_1, \alpha_2, \alpha_3)}(s_2, s_3)$ and $\Phi(z_1, z_2, z_3; s_2, s_3)$ are analytic on the domain $|s_2| < 1$, $|s_3| < 1$ and, respectively, for any z_1 , z_2 , and z_3 .

As in Theorem 3.1 of [9], we can show that

$$\Phi(z_1, z_2, z_3; s_2, s_3) = \lim_{t \rightarrow \infty} \mathcal{F}(t; z_1, z_2, z_3; s_1, s_2, s_3)$$

and $\Phi(z_1, z_2, z_3; s_2, s_3)$ satisfies the stationary first equation

$$\lambda_1 z_2 \left(\frac{\partial^2 \Phi}{\partial z_1 \partial z_3} - \frac{\partial^2 \Phi}{\partial z_1 \partial z_2} \right) + \lambda_2 z_3 \left(\frac{\partial \Phi}{\partial z_1} - \frac{\partial^2 \Phi}{\partial z_1 \partial z_3} \right) + \lambda_3 \left(\Phi - \frac{\partial \Phi}{\partial z_1} \right) = 0. \quad (3)$$

Let us obtain conditions on the function $\Phi(z_1, z_2, z_3; s_2, s_3)$. It follows from the equalities for the final probabilities

$$q_{(0, \alpha_2, \alpha_3)}^{(0, \alpha_2, \alpha_3)} = 1 \quad \text{and} \quad q_{(0, \gamma_2, \gamma_3)}^{(0, \alpha_2, \alpha_3)} = 0$$

for $\alpha_2 \neq \gamma_2$ or $\alpha_3 \neq \gamma_3$, that

$$\begin{aligned} \Phi(0, z_2, z_3; s_2, s_3) &= \sum_{\alpha_2, \alpha_3=0}^{\infty} \frac{z_2^{\alpha_2} z_3^{\alpha_3}}{\alpha_2! \alpha_3!} \Phi_{(0, \alpha_2, \alpha_3)}(s_2, s_3) \\ &= \sum_{\alpha_2, \alpha_3=0}^{\infty} \frac{z_2^{\alpha_2} z_3^{\alpha_3}}{\alpha_2! \alpha_3!} s_2^{\alpha_2} s_3^{\alpha_3} = e^{z_2 s_2 + z_3 s_3}. \end{aligned}$$

For the initial states $(\alpha_1, 0, 0)$, $\alpha_1 = 0, 1, 2, \dots$, the final probabilities are

$$q_{(0, 0, 0)}^{(\alpha_1, 0, 0)} = 1 \quad \text{and} \quad q_{(0, \gamma_2, \gamma_3)}^{(\alpha_1, 0, 0)} = 0$$

for $\gamma_2 \neq 0$ or $\gamma_3 \neq 0$. Therefore,

$$\Phi(z_1, 0, 0; s_2, s_3) = \sum_{\alpha_1=0}^{\infty} \frac{z_1^{\alpha_1}}{\alpha_1!} \Phi_{(\alpha_1, 0, 0)}(s_2, s_3) = \sum_{\alpha_2, \alpha_3=0}^{\infty} \frac{z_1^{\alpha_1}}{\alpha_1!} s_2^{\alpha_2} s_3^{\alpha_3} = e^{z_1}.$$

Thus, the linear partial differential equation (3) is considered under the boundary conditions

$$\Phi(0, z_2, z_3; s_1, s_2, s_3) = e^{z_2 s_2 + z_3 s_3}, \quad \Phi(z_1, 0, 0; s_1, s_2, s_3) = e^{z_1}. \quad (4)$$

In this paper, we obtain a solution of problem (3), (4) satisfying the analyticity condition. To study the questions of the existence and uniqueness of the solutions of equations of the form (3) is not the aim of the present paper.

4. INTEGRAL REPRESENTATION FOR THE FUNCTION $\Phi_{(\alpha_1, \alpha_2, \alpha_3)}(s_2, s_3)$

Theorem 1. Let $\rho = \lambda_2/\lambda_1$ and $\mu = \lambda_3/\lambda_1$; let $\rho \neq 1$. The exponential generating function of the final probabilities is

$$\begin{aligned} & \Phi(z_1, z_2, z_3; s_2, s_3) \\ &= \int_0^\infty \exp \left\{ -\mu v + z_2 \left(1 + (s_2 - 1)e^{-v} - (s_3 - 1) \frac{e^{-\rho v} - e^{-v}}{\rho - 1} \right) + z_3 (1 + (s_3 - 1)e^{-\rho v}) \right\} \\ & \quad \times \left[\mu + z_2 \left((s_2 - 1)e^{-v} - (s_3 - 1) \frac{\rho e^{-\rho v} - e^{-v}}{\rho - 1} \right) + z_3 (s_3 - 1) \rho e^{-\rho v} \right] \\ & \quad \times J_0(2\sqrt{-\mu z_1 v}) dv, \end{aligned} \quad (5)$$

where $J_0(z)$ is the Bessel function of order zero.

Proof. The function (5) is found as a solution of Eq. (3),

$$z_2 \frac{\partial^2 \Phi}{\partial z_1 \partial z_2} - (z_2 - \rho z_3) \frac{\partial^2 \Phi}{\partial z_1 \partial z_3} - (\rho z_3 - \mu) \frac{\partial \Phi}{\partial z_1} - \mu \Phi = 0. \quad (6)$$

Consider the change of variables $x = z_1$, $y = y(z_2, z_3)$, $\zeta = \zeta(z_2, z_3)$; we introduce the function $\Psi(x, y, \zeta; s_2, s_3)$ given by

$$\Phi(z_1, z_2, z_3; s_2, s_3) = \Psi(x, y(z_2, z_3), \zeta(z_2, z_3); s_2, s_3).$$

Then the principal part of Eq. (6) becomes

$$\begin{aligned} & z_2 \left(\frac{\partial^2 \Psi}{\partial x \partial y} \frac{\partial y}{\partial z_2} + \frac{\partial^2 \Psi}{\partial x \partial \zeta} \frac{\partial \zeta}{\partial z_2} \right) - (z_2 - \rho z_3) \left(\frac{\partial^2 \Psi}{\partial x \partial y} \frac{\partial y}{\partial z_3} + \frac{\partial^2 \Psi}{\partial x \partial \zeta} \frac{\partial \zeta}{\partial z_3} \right) \\ &= \frac{\partial^2 \Psi}{\partial x \partial \zeta} \left(z_2 \frac{\partial \zeta}{\partial z_2} - (z_2 - \rho z_3) \frac{\partial \zeta}{\partial z_3} \right) + \frac{\partial^2 \Psi}{\partial x \partial y} \left(z_2 \frac{\partial y}{\partial z_2} - (z_2 - \rho z_3) \frac{\partial y}{\partial z_3} \right). \end{aligned} \quad (7)$$

The first summand in (7) is eliminated because of the equality

$$z_2 \frac{\partial \zeta}{\partial z_2} - (z_2 - \rho z_3) \frac{\partial \zeta}{\partial z_3} = 0.$$

For the equation of first order, we obtain the first integral of the characteristic system and set

$$\zeta(z_2, z_3) = \exp\{(\rho - 1)z_2^{-\rho} z_3 - z_2^{1-\rho}\}.$$

The coefficient in the second summand in (7) can be simplified by an appropriate choice of the function $y(z_2, z_3)$. With regard to the second of the requirements (4), we impose the following limit condition on the change of variables:

$$z_2(y, \zeta) \rightarrow 0, \quad z_3(y, \zeta) \rightarrow 0 \quad \text{as } y \rightarrow 0.$$

This implies the study of the cases $\rho > 1$ and $0 < \rho < 1$ for which the choice of $y(z_2, z_3)$ is different.

(a) $\rho > 1$. The second summand in (7) can be simplified if the function $y(z_2, z_3)$ satisfies the equation

$$z_2 \frac{\partial y}{\partial z_2} + \rho z_3 \frac{\partial y}{\partial z_3} = 0.$$

We obtain the general solution of the last equation and take the function

$$y(z_2, z_3) = \exp\{-(\rho - 1)z_2^{-\rho} z_3\}.$$

We have the change of variables and the inverse change:

$$x = z_1, \quad y = \exp\{(1 - \rho)z_2^{-\rho} z_3\}, \quad \zeta = \exp\{(\rho - 1)z_2^{-\rho} z_3 - z_2^{1-\rho}\}; \quad (8)$$

$$z_1 = x, \quad z_2 = (-\ln \zeta y)^{1/(1-\rho)}, \quad z_3 = \frac{\ln y}{1-\rho} (-\ln \zeta y)^{\rho/(1-\rho)}. \quad (9)$$

The limit condition imposed above holds.

Substituting (9) into (6) and (4) and carrying out appropriate transformations, we obtain the equation

$$(\rho - 1)y \ln \zeta y \frac{\partial^2 \Psi}{\partial x \partial y} - \left(\frac{\rho}{\rho - 1} \ln y (-\ln \zeta y)^{\rho/(1-\rho)} + \mu \right) \frac{\partial \Psi}{\partial x} + \mu \Psi = 0 \quad (10)$$

with boundary conditions

$$\begin{aligned} \Psi(0, y, \zeta; s_2, s_3) &= \exp \left\{ (-\ln \zeta y)^{1/(1-\rho)} s_2 - \frac{\ln y}{\rho - 1} (-\ln \zeta y)^{\rho/(1-\rho)} s_3 \right\}, \\ \Psi(x, 0, \zeta; s_2, s_3) &= e^x. \end{aligned}$$

Further, the simplification of Eq. (10) is related to the summand with the first-order derivative: we make the standard substitution [12]

$$\Psi(x, y, \zeta; s_2, s_3) = u(x, y)g(y),$$

where $g(y)$ is a solution of the ordinary differential equation

$$(\rho - 1)y \ln \zeta y \frac{dg}{dy} - \left(\frac{\rho}{\rho - 1} \ln y (-\ln \zeta y)^{\rho/(1-\rho)} + \mu \right) g = 0.$$

We take the particular solution

$$g(y) = (-\ln \zeta y)^{-\mu/(1-\rho)} \exp \left\{ (-\ln \zeta y)^{1/(1-\rho)} - \frac{\ln y}{\rho - 1} (-\ln \zeta y)^{\rho/(1-\rho)} \right\}$$

and, after appropriate calculations, obtain the following equation for the function $u(x, y)$:

$$\frac{\partial^2 u}{\partial x \partial y} + \frac{\mu}{(\rho - 1)y \ln \zeta y} u = 0. \quad (11)$$

Here it is necessary to define and to solve an auxiliary boundary-value problem for Eq. (11). Suppose that $y_0 > 0$. We introduce the functions

$$\varphi(x) = \frac{\Psi(x, 0, \zeta; s_2, s_3)}{g(y_0)}, \quad \psi(y) = \frac{\Psi(0, y - y_0, \zeta; s_2, s_3)}{g(y)}.$$

Denote by $u^0(x, y)$ the solution of the hyperbolic equation (11) with boundary conditions on the characteristics $y = y_0$ and $x = 0$,

$$u^0(x, y_0) = \varphi(x), \quad u^0(0, y) = \psi(y). \quad (12)$$

Here

$$\varphi(0) = \lim_{y \rightarrow y_0} \psi(y_0) = \frac{1}{g(y_0)},$$

because $\lim_{y \rightarrow 0} \Psi(0, y, \zeta; s_2, s_3) = 1$ by the limit condition.

The Goursat problem (11), (12) is solved by using the Riemann method [12], [13]. The *Riemann function* $R(x, y; x_0, y_0)$ is defined as a solution of the equation conjugate to (11) (coinciding with (11) in our case)

$$\frac{\partial^2 R}{\partial x \partial y} + \frac{\mu}{(\rho - 1)y \ln \zeta y} R = 0, \quad (13)$$

satisfying the conditions

$$R(x, y_0; x_0, y_0) = \exp \int_{x_0}^x 0 d\tau = 1, \quad R(x_0, y; x_0, y_0) = \exp \int_{y_0}^y 0 dt = 1 \quad (14)$$

on the characteristics $y = y_0$ and $x = x_0$. By the substitution into Eq. (13) and verification of conditions (14), we establish that the the Riemann function is

$$R(x, y; x_0, y_0) = J_0 \left(2 \sqrt{-\frac{\mu}{\rho-1} (x-x_0) \ln \frac{\ln \zeta y_0}{\ln \zeta y}} \right). \quad (15)$$

Using the method given in [13], the function (15) can be found in the form of the series

$$R(x, y; x_0, y_0) = \sum_{j=0}^{\infty} \frac{v_j(x, y; x_0, y_0)(x-x_0)^j(y-y_0)^j}{j! j!}. \quad (16)$$

The series (16) is next substituted into Eq. (13) and the recurrence differential relation is solved for the functions $v_j(x, y; x_0, y_0)$ (see the proof of Lemma 1 in [11]).

The solution of problem (11), (12) is given by the Riemann formula ($x > 0, y > y_0$)

$$\begin{aligned} u^0(x, y) &= R(x, y_0; x, y)\varphi(x) + R(0, y; x, y)\psi(y) - R(0, y_0; x, y)\varphi(0) \\ &\quad - \int_0^x \frac{\partial R(\tau, y_0; x, y)}{\partial \tau} \varphi(\tau) d\tau - \int_{y_0}^y \frac{\partial R(0, t; x, y)}{\partial t} \psi(t) dt; \end{aligned}$$

integrating by parts, we obtain

$$u^0(x, y) = \frac{R(0, y_0; x, y)}{g(y_0)} + \int_0^x \frac{R(\tau, y_0; x, y)e^\tau}{g(y_0)} d\tau + \int_{y_0}^y \frac{\partial \psi(t)}{\partial t} R(0, t; x, y) dt. \quad (17)$$

Now let us pass to the limit as $y_0 \rightarrow 0$. The explicit form of the functions (15) and $g(y)$ implies

$$\lim_{y_0 \rightarrow 0} \frac{R(0, y_0; x, y)}{g(y_0)} = 0, \quad \lim_{y_0 \rightarrow 0} \frac{R(\tau, y_0; x, y)e^\tau}{g(y_0)} = 0.$$

Accordingly, as $y_0 \rightarrow 0$, expression (17) for $u^0(x, y)$ yields the following solution of Eq. (11):

$$u(x, y) = \int_0^y \frac{\partial \theta(t)}{\partial t} R(0, t; x, y) dt;$$

here

$$\theta(t) = \lim_{y_0 \rightarrow 0} \psi(t) = (-\ln \zeta t)^{\mu/(1-\rho)} \exp \left\{ (s_2 - 1)(-\ln \zeta t)^{1/(1-\rho)} - (s_3 - 1) \frac{\ln t}{\rho-1} (-\ln \zeta t)^{\rho/(1-\rho)} \right\}.$$

After the substitution, we have

$$\begin{aligned} u(x, y) &= \int_0^y \frac{(-\ln \zeta t)^{\mu/(1-\rho)-1}}{(\rho-1)t} \exp \left\{ (s_2 - 1)(-\ln \zeta t)^{1/(1-\rho)} - (s_3 - 1) \frac{\ln t}{\rho-1} (-\ln \zeta t)^{\rho/(1-\rho)} \right\} \\ &\quad \times \left[\mu + (s_2 - 1)(-\ln \zeta t)^{1/(1-\rho)} - (s_3 - 1) \left((-\ln \zeta t)^{1/(1-\rho)} + \frac{\rho \ln t}{\rho-1} (-\ln \zeta t)^{\rho/(1-\rho)} \right) \right] \\ &\quad \times J_0 \left(2 \sqrt{-\frac{\mu}{\rho-1} x \ln \frac{\ln \zeta t}{\ln \zeta y}} \right) dt \end{aligned}$$

(the convergence of the improper integral is obvious; $\rho > 1$). After the change of the variable of integration $v = \ln(\ln \zeta t / \ln \zeta y) / (\rho-1)$ (i.e., $\ln \zeta t = e^{(\rho-1)v} \ln \zeta y$, $dt/t = (\rho-1)e^{(\rho-1)v} \ln \zeta y dv$), we

obtain

$$\begin{aligned}
u(x, y) = & \int_0^\infty e^{-\mu v} (-\ln \zeta y)^{\mu/(1-\rho)} \exp \left\{ (s_2 - 1)e^{-v} (-\ln \zeta y)^{1/(1-\rho)} \right. \\
& \quad \left. - (s_3 - 1) \frac{e^{(\rho-1)v} \ln \zeta y - \ln \zeta}{\rho - 1} e^{-\rho v} (-\ln \zeta y)^{\rho/(1-\rho)} \right\} \\
& \times \left[\mu + (s_2 - 1)e^{-v} (-\ln \zeta y)^{1/(1-\rho)} - (s_3 - 1) \left(e^{-v} (-\ln \zeta y)^{1/(1-\rho)} \right. \right. \\
& \quad \left. \left. + \frac{\rho(e^{(\rho-1)v} \ln \zeta y - \ln \zeta)}{\rho - 1} e^{-\rho v} (-\ln \zeta y)^{\rho/(1-\rho)} \right) \right] J_0(2\sqrt{-\mu xv}) dv.
\end{aligned}$$

Further, $\Psi(x, y, \zeta; s_2, s_3) = u(x, y)g(y)$ and, returning from the variables x, y, ζ to the variables z_1, z_2, z_3 according to (8), we obtain expression (5).

The passage to the limit as $y_0 \rightarrow 0$ requires justification; however, substituting expression (5) directly into Eq. (6) and verifying conditions (4), we see that (5) is a solution of problem (6), (4). The analyticity requirement of the solution by the variables z_1, z_2, z_3 is met, while the uniqueness of the solution (5) is implied by the fact that the solution is analytic (see [13], [3]).

(b) $0 < \rho < 1$. We use the change of variables

$$\begin{aligned}
x = z_1, \quad y = \exp \left\{ -(\rho - 1)z_2^{-\rho} z_3 + z_2^{1-\rho} + \frac{\rho - 1}{\rho} z_2^{-\rho} \right\}, \quad \zeta = \exp \{(\rho - 1)z_2^{-\rho} z_3 - z_2^{1-\rho}\}; \\
z_1 = x, \quad z_2 = \left(\frac{\rho}{\rho - 1} \ln \zeta y \right)^{-1/\rho}, \quad z_3 = \frac{1}{\rho \ln \zeta y} \left(\ln \zeta + \left(\frac{\rho}{\rho - 1} \ln \zeta y \right)^{(\rho-1)/\rho} \right).
\end{aligned}$$

The following limit condition related to (4) holds:

$$z_2(y, \zeta) \rightarrow 0, \quad z_3(y, \zeta) \rightarrow 0 \quad \text{as } y \rightarrow 0$$

After the substitution into (6) and (4), we obtain the equation

$$\Psi(x, y, \zeta; s_2, s_3) = \rho y \ln \zeta y \frac{\partial^2 \Psi}{\partial x \partial y} - \left(\frac{1}{\ln \zeta y} \left(\ln \zeta + \left(\frac{\rho}{\rho - 1} \ln \zeta y \right)^{(\rho-1)/\rho} \right) - \mu \right) \frac{\partial \Psi}{\partial x} - \mu \Psi = 0$$

with boundary conditions

$$\begin{aligned}
\Psi(0, y, \zeta; s_2, s_3) &= \exp \left\{ \left(\frac{\rho \ln \zeta y}{(\rho - 1)} \right)^{-1/\rho} s_2 + \left(\ln \zeta + \left(\frac{\rho}{\rho - 1} \right) \ln \zeta y \right)^{(\rho-1)/\rho} / (\rho \ln \zeta y) s_3 \right\}, \\
\Psi(x, 0, \zeta; s_2, s_3) &= e^x.
\end{aligned}$$

Further, the solution (5) is obtained just as in case (a). The expression for the Riemann function coincides with (15) (instead of $\rho - 1$ we have ρ). Theorem 1 is proved. \square

Corollary 1. *For the Markov process $\xi(t)$, the generating function of the final probabilities ($\rho \neq 1$, $\alpha_1 \neq 0$) is as follows:*

$$\begin{aligned}
\Phi_{(\alpha_1, \alpha_2, \alpha_3)}(s_2, s_3) = & \frac{\mu^{\alpha_1}}{(\alpha_1 - 1)!} \int_0^\infty v^{\alpha_1 - 1} e^{-\mu v} \\
& \times \left(1 + (s_2 - 1)e^{-v} - (s_3 - 1) \frac{e^{-\rho v} - e^{-v}}{\rho - 1} \right)^{\alpha_2} (1 + (s_3 - 1)e^{-\rho v})^{\alpha_3} dv. \quad (18)
\end{aligned}$$

Proof. The generating function (2) can be expressed as

$$\begin{aligned}\Phi(z_1, z_2, z_3; s_2, s_3) &= \sum_{\alpha_1=0}^{\infty} \frac{z_1^{\alpha_1}}{\alpha_1!} \Phi_{(\alpha_1, 0, 0)}(s_2, s_3) + \sum_{\alpha_1=0}^{\infty} \sum_{\alpha_2=1}^{\infty} \frac{z_1^{\alpha_1} z_2^{\alpha_2}}{\alpha_1! \alpha_2!} \Phi_{(\alpha_1, \alpha_2, 0)}(s_2, s_3) \\ &\quad + \sum_{\alpha_2=0}^{\infty} \sum_{\alpha_3=1}^{\infty} \frac{z_2^{\alpha_2} z_3^{\alpha_3}}{\alpha_2! \alpha_3!} \Phi_{(0, \alpha_2, \alpha_3)}(s_2, s_3) \\ &\quad + \sum_{\alpha_1=1}^{\infty} \sum_{\alpha_2=0}^{\infty} \sum_{\alpha_3=1}^{\infty} \frac{z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3}}{\alpha_1! \alpha_2! \alpha_3!} \Phi_{(\alpha_1, \alpha_2, \alpha_3)}(s_2, s_3).\end{aligned}\tag{19}$$

To reduce expression (5) to a similar sum, we use the expansions

$$J_0(z) = \sum_{i=0}^{\infty} (-1)^i \frac{(z/2)^{2i}}{(i! i!)^2} \quad \text{and} \quad e^{az_3}(b + cz_3) = b + \sum_{j=1}^{\infty} \left(\frac{z_3^j}{j!} \right) (ba^j + jca^{j-1});$$

it then follows from (5) that

$$\begin{aligned}\Phi(z_1, z_2, z_3; s_2, s_3) &= \sum_{\alpha_1=0}^{\infty} \frac{\mu^{\alpha_1} z_1^{\alpha_1}}{\alpha_1! \alpha_1!} \int_0^{\infty} v^{\alpha_1} \exp \left\{ -\mu v + z_2 \left(1 + (s_2 - 1)e^{-v} - (s_3 - 1) \frac{e^{-\rho v} - e^{-v}}{\rho - 1} \right) \right\} \\ &\quad \times \left[\mu + z_2 \left((s_2 - 1)e^{-v} - (s_3 - 1) \frac{\rho e^{-\rho v} - e^{-v}}{\rho - 1} \right) \right] dv \\ &\quad + \sum_{\alpha_1=0}^{\infty} \sum_{\alpha_3=1}^{\infty} \frac{\mu^{\alpha_1} z_1^{\alpha_1} z_3^{\alpha_3}}{\alpha_1! \alpha_1! \alpha_3!} \\ &\quad \times \int_0^{\infty} v^{\alpha_1} \exp \left\{ -\mu v + z_2 \left(1 + (s_2 - 1)e^{-v} - (s_3 - 1) \frac{e^{-\rho v} - e^{-v}}{\rho - 1} \right) \right\} \\ &\quad \times \left[\left(\mu + z_2 \left((s_2 - 1)e^{-v} - (s_3 - 1) \frac{\rho e^{-\rho v} - e^{-v}}{\rho - 1} \right) \right) \right. \\ &\quad \left. \times (1 + (s_3 - 1)e^{-\rho v})^{\alpha_3} + \alpha_3 (s_3 - 1) \rho e^{-\rho v} (1 + (s_3 - 1)e^{-\rho v})^{\alpha_3-1} \right] dv.\end{aligned}\tag{20}$$

In the first summand, we substitute the expansion

$$e^{az_2}(b + cz_2) = b + \sum_{k=1}^{\infty} \frac{z_2^k}{k!} (ba^k + kca^{k-1});$$

further, in the resulting expression, the integral is split into two integrals and the second of them is integrated by parts. In the second summand in (20), the integral is split into two integrals and the second of them is integrated by parts (taking into account the cases $\alpha_1 = 0$ and $\alpha_1 > 0$); further, in the resulting expression, we substitute the expansion

$$e^{az_2} = \sum_{k=0}^{\infty} \frac{z_2^k}{k!} a^k.$$

As a result of the transformations, from (20) we obtain

$$\begin{aligned}\Phi(z_1, z_2, z_3; s_2, s_3) &= e^{z_1} + e^{z_2 s_2 + z_3 s_3} - 1 + \sum_{\alpha_1=1}^{\infty} \sum_{\alpha_2=1}^{\infty} \frac{\mu^{\alpha_1} z_1^{\alpha_1} z_2^{\alpha_2}}{\alpha_1! (\alpha_1 - 1)! \alpha_2!} \\ &\quad \times \int_0^{\infty} v^{\alpha_1-1} e^{-\mu v} \left(1 + (s_2 - 1)e^{-v} - (s_3 - 1) \frac{e^{-\rho v} - e^{-v}}{\rho - 1} \right)^{\alpha_2} dv\end{aligned}$$

$$\begin{aligned}
& + \sum_{\alpha_1=1}^{\infty} \sum_{\alpha_2=0}^{\infty} \sum_{\alpha_3=1}^{\infty} \frac{\mu^{\alpha_1} z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3}}{\alpha_1! (\alpha_1 - 1)! \alpha_2! \alpha_3!} \\
& \quad \times \int_0^{\infty} v^{\alpha_1-1} e^{-\mu v} \left(1 + (s_2 - 1)e^{-v} - (s_3 - 1) \frac{e^{-\rho v} - e^{-v}}{\rho - 1} \right)^{\alpha_2} \\
& \quad \times (1 + (s_3 - 1)e^{-v})^{\alpha_3} dv.
\end{aligned}$$

In the resulting series and the series (19), equating the coefficients of the identical powers of z_1, z_2, z_3 , we obtain the integral representation (18). Corollary 1 is proved. \square

5. THE ASYMPTOTIC PROPERTIES OF THE FINAL DISTRIBUTION

In the special case under consideration, particles of types T_2 and T_3 are said to be *final* [10]. Denote by $\eta_2^{(\alpha_1, \alpha_2, \alpha_3)}$ the random number of particles of type T_2 and by $\eta_3^{(\alpha_1, \alpha_2, \alpha_3)}$ the random number of particles of type T_2 that remain after the termination of the epidemic process, i.e., there no longer remain any particles of type T_1 . The random vector $(\eta_2^{(\alpha_1, \alpha_2, \alpha_3)}, \eta_3^{(\alpha_1, \alpha_2, \alpha_3)})$ has the probability distribution

$$\{q_{(0\gamma_2, \gamma_3)}^{(\alpha_1, \alpha_2, \alpha_3)}, (\gamma_2 \gamma_3) \in \mathbb{N}^2\},$$

which is defined by the generating function (18). For the expectations, as $\alpha_2 \rightarrow \infty$, we obtain

$$\begin{aligned}
E\eta_2^{(\alpha_1, \alpha_2, \alpha_3)} &= \frac{\partial \Phi_{(\alpha_1, \alpha_2, \alpha_3)}(1, 1)}{\partial s_2} = \alpha_2 \left(\frac{\mu}{\mu + 1} \right)^{\alpha_1}, \\
E\eta_3^{(\alpha_1, \alpha_2, \alpha_3)} &= \frac{\partial \Phi_{(\alpha_1, \alpha_2, \alpha_3)}(1, 1)}{\partial s_3} \sim \frac{\alpha_2}{\rho - 1} \left(\left(\frac{\mu}{\mu + 1} \right)^{\alpha_1} - \left(\frac{\mu}{\mu + \rho} \right)^{\alpha_1} \right).
\end{aligned}$$

The calculation of the variances leads to the following asymptotic formulas as $\alpha_2 \rightarrow \infty$:

$$\begin{aligned}
D\eta_2^{(\alpha_1, \alpha_2, \alpha_3)} &= \frac{\partial^2 \Phi_{(\alpha_1, \alpha_2, \alpha_3)}(1, 1)}{\partial s_2^2} + \frac{\partial \Phi_{(\alpha_1, \alpha_2, \alpha_3)}(1, 1)}{\partial s_2} - \left(\frac{\partial \Phi_{(\alpha_1, \alpha_2, \alpha_3)}(1, 1)}{\partial s_2} \right)^2 \\
&\sim \alpha_2^2 \left(\left(\frac{\mu}{\mu + 2} \right)^{\alpha_1} - \left(\frac{\mu}{\mu + 1} \right)^{2\alpha_1} \right), \\
D\eta_3^{(\alpha_1, \alpha_2, \alpha_3)} &\sim \frac{\alpha_2^2}{(\rho - 1)^2} \left(\left(\frac{\mu}{\mu + 2} \right)^{\alpha_1} - 2 \left(\frac{\mu}{\mu + \rho + 1} \right)^{\alpha_1} + \left(\frac{\mu}{\mu + 2\rho} \right)^{\alpha_1} \right. \\
&\quad \left. - \left(\frac{\mu}{\mu + 1} \right)^{2\alpha_1} + 2 \left(\frac{\mu^2}{(\mu + 1)(\mu + \rho)} \right)^{\alpha_1} - \left(\frac{\mu}{\mu + \rho} \right)^{2\alpha_1} \right).
\end{aligned}$$

Using the explicit expression (18) for the generating function of the probability distribution on \mathbb{N}^2 and applying the Laplace transform to derive the limit theorems [14], [10] in the standard way, we obtain the following statement.

Theorem 2. Suppose that $x_1, x_2 \in [0, 1]$. Then ($\rho \neq 1, \alpha_1 \neq 0$)

$$\lim_{\alpha_2 \rightarrow \infty} P \left\{ \frac{\eta_2^{(\alpha_1, \alpha_2, \alpha_3)}}{\alpha_2} \leq x_1, \frac{\eta_3^{(\alpha_1, \alpha_2, \alpha_3)}}{\alpha_2} \leq x_2 \right\} = \int_0^{x_1} \int_0^{x_2} f(y_1, y_2) dy_1 dy_2;$$

the two-dimensional Laplace transform for the probability distribution density $f(x_1, x_2)$ is of the form ($\lambda_1 \geq 0, \lambda_2 \geq 0$)

$$\begin{aligned}
F(\lambda_1, \lambda_2) &= \int_0^{\infty} \int_0^{\infty} e^{-\lambda_1 x_1 - \lambda_2 x_2} f(x_1, x_2) dx_1 dx_2 \\
&= \frac{\mu^{\alpha_1}}{(\alpha_1 - 1)!} \int_0^{\infty} v^{\alpha_1-1} e^{-\mu v - \lambda_1 e^{-v} + \lambda_2 (e^{-\rho v} - e^{-v})/(\rho - 1)} dv. \tag{21}
\end{aligned}$$

Remark. The two-dimensional distribution density can be expressed as

$$f(x_1, x_2) = e^{-(x_1+x_2)/2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_{nm}}{n! m!} L_n(x_1) L_m(x_2), \quad x_1 \geq 0, \quad x_2 \geq 0, \quad (22)$$

where $L_n(x)$, $n = 0, 1, 2, \dots$, is the Laguerre polynomial and

$$a_{nm} = \frac{\mu^{\alpha_1}}{(\alpha_1 - 1)!} \sum_{i=0}^n \sum_{j=0}^m \frac{\binom{n}{i} \binom{m}{j}}{i! j!} \int_0^\infty v^{\alpha_1-1} \\ \times (-1)^i \left(\frac{e^{-\rho v} - e^{-v}}{\rho - 1} \right)^j e^{-(i+\mu)v - (1/2)e^{-v} + (1/2)(e^{-\rho v} - e^{-v})/(\rho-1)} dv.$$

The expansion in Laguerre polynomials is standard in operator calculus. Expression (22) can be verified by directly calculating the Laplace transform (22) taking into account the equality ($\lambda \geq 0$)

$$\int_0^\infty e^{-(\lambda+1/2)x} L_n(x) dx = \frac{n!}{\lambda + 1/2} \left(\frac{\lambda - 1/2}{\lambda + 1/2} \right)^n, \quad n = 0, 1, 2, \dots,$$

and using the Taylor series expansion ($|p_1| < 1$, $|p_2| < 1$)

$$\frac{F((1/2)(1+p_1)/(1-p_1), (1/2)(1+p_2)/(1-p_2))}{(1-p_1)(1-p_2)} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{nm} p_1^n p_2^m, \\ a_{nm} = \sum_{i=0}^n \sum_{j=0}^m \frac{\binom{n}{i} \binom{m}{j}}{i! j!} \frac{\partial^{i+j} F(1/2, 1/2)}{\partial \lambda_1^i \partial \lambda_2^j}.$$

The random variables $\eta_2^{(\alpha_1, \alpha_2, \alpha_3)}$ and $\eta_3^{(\alpha_1, \alpha_2, \alpha_3)}$ have the distributions defined by the generating functions $\Phi_{(\alpha_1, \alpha_2, \alpha_3)}(s_2, 1)$ and $\Phi_{(\alpha_1, \alpha_2, \alpha_3)}(1, s_3)$. Accordingly, by setting $\lambda_2 = 0$ or $\lambda_1 = 0$ in (21), we obtain corollaries for the one-dimensional distributions.

Corollary 2 ([3]). Suppose that $x \in [0, 1]$. Then

$$\lim_{\alpha_2 \rightarrow \infty} P \left\{ \frac{\eta_2^{(\alpha_1, \alpha_2, \alpha_3)}}{\alpha_2} \leq x \right\} = \frac{\mu^{\alpha_1}}{(\alpha_1 - 1)!} \int_{-\ln x}^{\infty} y^{\alpha_1-1} e^{-\mu y} dy.$$

Consider the function

$$x(y) = -(e^{-\rho y} - e^{-y})/(\rho - 1), \quad y \in [0, \infty).$$

The function $x(y)$ is increasing on $[0, y_0]$ and decreasing on $[y_0, \infty)$, y_0 is the point of maximum, and $x_0 = x(y_0) \leq 1$. On the closed interval $[0, x_0]$, the inverse functions $y_1(x)$, $y_1(0) = 0$, and $y_2(x)$, $\lim_{x \rightarrow 0} y_2(x) = \infty$, are defined; moreover, $y_1(x_0) = y_2(x_0)$.

Corollary 3. Suppose that $x \in [0, x_0]$. Then

$$\lim_{\alpha_2 \rightarrow \infty} P \left\{ \frac{\eta_3^{(\alpha_1, \alpha_2, \alpha_3)}}{\alpha_2} \leq x \right\} = \frac{\mu^{\alpha_1}}{(\alpha_1 - 1)!} \left(\int_0^{y_1(x)} y^{\alpha_1-1} e^{-\mu y} dy + \int_{y_2(x)}^{\infty} y^{\alpha_1-1} e^{-\mu y} dy \right).$$

Consider the case of the coordinated convergence of α_2 and α_3 to infinity. Relation (18) implies the following limit theorem.

Theorem 3. Suppose that $\alpha_2 \rightarrow \infty$, $\alpha_3 \rightarrow \infty$, and the ratio α_3/α_2 tends to \varkappa , where $0 \leq \varkappa < \infty$. Suppose that $x_1, x_2 \in [0, 1]$. Then ($\rho \neq 1$, $\alpha_1 \neq 0$)

$$\lim_{\substack{\alpha_2 \rightarrow \infty \\ \alpha_3 \rightarrow \infty}} P \left\{ \frac{\eta_2^{(\alpha_1, \alpha_2, \alpha_3)}}{\alpha_2} \leq x_1, \frac{\eta_3^{(\alpha_1, \alpha_2, \alpha_3)}}{\alpha_2} \leq x_2 \right\} = \int_0^{x_1} \int_0^{x_2} f(y_1, y_2) dy_1 dy_2;$$

the Laplace transform for the probability distribution density is of the form ($\lambda_1 \geq 0, \lambda_2 \geq 0$)

$$\begin{aligned} F(\lambda_1, \lambda_2) &= \int_0^\infty \int_0^\infty e^{-\lambda_1 x_1 - \lambda_2 x_2} f(x_1, x_2) dx_1 dx_2 \\ &= \frac{\mu^{\alpha_1}}{(\alpha_1 - 1)!} \int_0^\infty v^{\alpha_1 - 1} e^{-\mu v - \lambda_1 e^{-v} + \lambda_2 (e^{-\rho v} (1 - \varkappa(\rho - 1)) - e^{-v}) / (\rho - 1)} dv. \end{aligned}$$

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