Journal of Mathematical Sciences, Vol. 99, No. 3, 2000

BRANCHING PROPERTY FOR A POISSON-TYPE DEATH PROCESS

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UDC 519.2

A generalized branching property for the transition probabilities of a Poisson-type pure death process is obtained.

1. Definition of a Pure Death Process

Let ξ_t, $t \in [0, \infty)$, be a time-homogeneous Markov process with state space $\{0, 1, 2, \dots\}$. Denote by

$$P_{ij}(t) = \mathbf{P}\{\xi_t = j \mid \xi_0 = i\}, \quad i, j = 0, 1, \dots,$$

the transition probabilities of ξ_t. Assume that

$$P_{i,i-1}(t) = \mu_i t + o(t), \qquad P_{ii}(t) = 1 - \mu_i t + o(t) \quad \text{as} \quad t \to 0,$$

where $\mu_0 = 0$, $\mu_i \geq 0$, $i = 1, 2, \dots$.

For this pure death process ξ_t, the exact formulas for the transition probabilities are known:

$$P_{0j}(t) = \delta_j^0, \quad j = 0, 1, \dots, \qquad P_{ij}(t) = 0, \quad j > i \geq 1,$$

$$P_{ij}(t) = \mu_i \cdots \mu_{j+1} \sum_{k=j}^{i} \frac{e^{-\mu_k t}}{(\mu_i - \mu_k) \cdots (\mu_{k+1} - \mu_k)(\mu_{k-1} - \mu_k) \cdots (\mu_j - \mu_k)}, \quad j \leq i.$$

Here and in what follows, we use the Kronecker symbol

$$\delta_j^i = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

2. Kolmogorov Equations for the Generating Function of the Transition Probabilities

Introduce the generating functions of the transition probabilities:

$$F_i(t; s) = \sum_{j=0}^{\infty} P_{ij}(t) s^j, \quad i = 0, 1, \dots, \quad |s| \leq 1.$$

Assume that $f(s)$ is analytic at 0:

$$f(s) = \sum_{j=0}^{\infty} a_j s^j.$$

Consider the Gelfond–Leontiev operator of the generalized derivative:

$$D_s(f) = \sum_{j=1}^{\infty} a_j \mu_j s^{j-1}.$$

The second (forward) system of Kolmogorov differential equations for the transition probabilities $P_{ij}(t)$ of ξ_t can be written more compactly by means of the generating functions and the operator of the generalized derivative:

$$\frac{\partial F_i}{\partial t} = (1 - s) D_s(F_i),$$

with boundary condition $F_i(0, s) = s^i$.

Indeed, the second Kolmogorov system of equations for a pure death process has the form

$$\frac{dP_{i0}(t)}{dt} = P_{i1}(t)\mu_1, \qquad \frac{dP_{ij}(t)}{dt} = -P_{ij}(t)\mu_j + P_{i,j+1}(t)\mu_{j+1}, \quad j = 1, 2 \dots .$$

Proceedings of the Seminar on Stability Problems for Stochastic Models, Vologda, Russia, 1998, Part I.

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We have the chain of equalities

$$\frac{\partial F_i}{\partial t} = \sum_{j=0}^{\infty} \frac{dP_{ij}(t)}{dt} s^j = -\sum_{j=1}^{\infty} P_{ij}(t)\mu_j s^j + \sum_{j=0}^{\infty} P_{i,j+1}(t)\mu_{j+1} s^j$$

$$= -s\sum_{j=1}^{\infty} P_{ij}(t)\mu_j s^{j-1} + \sum_{j=1}^{\infty} P_{ij}(t)\mu_j s^{j-1} = -sD_s(F_i) + D_s(F_i) = (1-s)D_s(F_i).$$

3. Branching Property for a Linear Death Process

In the case of a linear death process, i.e., when $\mu_i = i\mu$ $(\mu > 0)$,

$$D_s = \mu \frac{d}{ds},$$

we have the partial differential equation ([1, Chap. 1, §4, Theorem 5])

$$\frac{\partial F_i(t;s)}{\partial t} = \mu(1-s)\frac{\partial F_i(t;s)}{\partial s}$$

with boundary condition $F_i(0,s) = s^i$.

In the linear case, the exact formulas for the transition probabilities have the form

$$P_{ij}(t) = 0, \quad j > i,$$

$$P_{ij}(t) = C_i^j (1-e^{-\mu t})^{i-j}(e^{-\mu t})^j, \quad j \le i.$$

Here, the branching property takes place:

$$F_i(t;s) = F_1^i(t;s), \quad i = 0, 1, \ldots$$

(see [1, Chap. 1, §1, (5)]), where

$$F_1(t;s) = 1 - e^{-\mu t} + se^{-\mu t}$$

(see [1, Chap. 1, §8, Ex. 2]).

4. The Results

For a Poisson-type death process, i.e., when $\mu_0 = 0$, $\mu_i = \mu$, $i = 1, 2, \ldots$,

$$D_s(f) = \mu \frac{f(s) - f(0)}{s},$$

we have (see [6])

$$\frac{\partial F_i(t,s)}{\partial t} = \mu(1-s)\frac{F_i(t;s) - F_i(t;0)}{s}$$

with boundary condition $F_i(0,s) = s^i$.

In the Poisson case, the exact formulas for the transition probabilities $P_{ij}(t)$ are

$$P_{00}(t) = 1, \qquad P_{i0}(t) = 1 - e^{-\mu t} - \frac{\mu t}{1!}e^{-\mu t} - \cdots - \frac{(\mu t)^{i-1}}{(i-1)!}e^{-\mu t}, \quad i = 1, 2, \ldots,$$

$$P_{ij}(t) = \frac{(\mu t)^{i-j}}{(i-j)!}e^{-\mu t}, \quad 1 \le j \le i, \qquad P_{ij}(t) = 0, \quad j > i. \tag{1}$$

Theorem 1. *For a Poisson-type death process* ξ_t $(i = 0, 1, \ldots; \omega^2 = -1)$

$$F_i(t;s) = 1 + \int_0^1 \left(\frac{1}{2\pi\omega} \oint_{0+} \left[\int_0^{\infty} [\varphi^i(x,\eta,u;s)]'_\eta \, d\eta \right]'_x \frac{e^{\mu t(ux-1)}}{u} \, du \right) dx, \tag{2}$$

where

$$\varphi(x,\eta,u;s) = \frac{1}{u} + \eta\frac{1-x}{1+\eta} + s\frac{1-x}{1+\eta}$$

is a linear function in s.

In (2) and in the arguments below, we use the following notation for a function of several variables:

$$[H(x,u;s)]'_x = \frac{\partial H(x,u;s)}{\partial x}.$$

The proof of Theorem 1 is based on direct calculations of transition probabilities (1). Indeed, we note that (2) is equivalent to the equality

$$P_{ij}(t) = \delta_j^0 + \int_0^1 \left(\frac{1}{2\pi\omega} \oint_{0+} \left[\int_0^\infty \left[C_i^j \left(\frac{1}{u} + \eta\frac{1-x}{1+\eta} \right)^{i-j} \left(\frac{1-x}{1+\eta} \right)^j \right]'_\eta d\eta \right]'_x \frac{e^{\mu t(ux-1)}}{u}\, du \right) dx, \quad j \le i.$$

Calculating the integral above, for $j \ge 1$ we obtain

$$P_{ij}(t) = \int_0^1 \left(\frac{1}{2\pi\omega} \oint_{0+} \left[-C_i^j \left(\frac{1}{u} \right)^{i-j} (1-x)^j \right]'_x \frac{e^{\mu t(ux-1)}}{u}\, du \right) dx$$

$$= e^{-\mu t} \int_0^1 \left(\frac{1}{2\pi\omega} \oint_{0+} jC_i^j (1-x)^{j-1} \frac{e^{\mu tux}}{u^{i-j+1}}\, du \right) dx = e^{-\mu t} \int_0^1 jC_i^j (1-x)^{j-1} \frac{(\mu tx)^{i-j}}{(i-j)!}\, dx$$

$$= e^{-\mu t} jC_i^j B(i-j+1, j) \frac{(\mu t)^{i-j}}{(i-j)!} = \frac{(\mu t)^{i-j}}{(i-j)!} e^{-\mu t}.$$

Here, for integer positive n and m

$$B(n,m) = \int_0^1 x^{n-1}(1-x)^{m-1}\, dx = \frac{(n-1)!\,(m-1)!}{(n+m-1)!}$$

is the beta-function. Calculate $P_{i0}(t)$ for $i \ge 1$:

$$P_{i0}(t) = 1 + \int_0^1 \left(\frac{1}{2\pi\omega} \oint_{0+} \left[\int_0^\infty \left[\left(\frac{1}{u} + \eta\frac{1-x}{1+\eta} \right)^i \right]'_\eta d\eta \right]'_x \frac{e^{\mu t(ux-1)}}{u}\, du \right) dx$$

$$= 1 + \int_0^1 \left(\frac{1}{2\pi\omega} \oint_{0+} \left[\left(\frac{1}{u} + 1 - x \right)^i - \left(\frac{1}{u} \right)^i \right]'_x \frac{e^{\mu t(ux-1)}}{u}\, du \right) dx$$

$$= 1 - \int_0^1 \left(\frac{1}{2\pi\omega} \oint_{0+} i \left(\frac{1}{u} + 1 - x \right)^{i-1} \frac{e^{\mu t(ux-1)}}{u}\, du \right) dx$$

$$= 1 - e^{-\mu t} \int_0^1 \left(\frac{1}{2\pi\omega} \oint_{0+} i \left(\sum_{k=0}^{i-1} C_{i-1}^k \left(\frac{1}{u} \right)^k (1-x)^{i-1-k} \right) \frac{e^{\mu tux}}{u}\, du \right) dx$$

$$= 1 - e^{-\mu t} \int_0^1 i \left(\sum_{k=0}^{i-1} C_{i-1}^k (1-x)^{i-1-k} \frac{(\mu tx)^k}{k!} \right) dx$$

$$= 1 - e^{-\mu t} \sum_{k=0}^{i-1} i C_{i-1}^{k} B(k+1, i-k) \frac{(\mu t)^k}{k!} = 1 - e^{-\mu t} - \frac{\mu t}{1!} e^{-\mu t} - \dots - \frac{(\mu t)^{i-1}}{(i-1)!} e^{-\mu t}.$$

Calculation of $P_{00}(t)$ gives us the value 1.

To obtain (2) one can consider the exponential (double) generating function (see [3]) of the transition probabilities $P_{ij}(t)$ and use a special summation formula of the Bessel function theory [9].

5. Branching Property for a Square-Type Death Process

We recall the main results of [4] and [10].

In the case of a square-type death process, i.e., when $\mu_i = i(i-1)\mu$,

$$D_s = \mu s \frac{d^2}{ds^2},$$

we have the following second-order partial differential equation (see [2; 3, Theorem 1; [5]):

$$\frac{\partial F_i(t;s)}{\partial t} = \mu(s - s^2) \frac{\partial^2 F_i(t;s)}{\partial s^2}, \tag{3}$$

with boundary condition $F_i(0,s) = s^i$.

In the square case, the exact formulas for the transition probabilities $P_{ij}(t)$ have the form

$$P_{ij}(t) = \delta_j^i, \quad i = 0, 1, \quad j = 0, 1, \dots, \qquad P_{ij}(t) = 0, \quad j > i \geq 2, \qquad P_{i0}(t) = \delta_0^i,$$

$$P_{ij}(t) = \frac{(i-1)!\, i!}{(j-1)!\, j!} \sum_{k=j}^{i} \frac{(-1)^{k-j}(2k-1)(k+j-2)!}{(i-k)!\,(i+k-1)!\,(k-j)!} e^{-k(k-1)\mu t}, \quad 1 \leq j \leq i. \tag{4}$$

The integral representation for the transition probabilities $P_{ij}(t)$ given in [4] is analogous to representation (2).

THEOREM 2. *For a square-type pure death process ξ_t $(i = 0, 1, \dots; \omega^2 = -1)$ the following relation is valid:*

$$F_i(t;s) = 1 + \frac{e^{\mu t/4}}{2\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-v^2/(\mu t)} \left[\int_{\cos 2v}^{1} \frac{1}{\sqrt{y - \cos 2v}} \left(\frac{1}{2\pi\omega} \oint_{0+} \varphi^i(\eta, y; s) \frac{d\eta}{\sqrt{1 - 2\eta y + \eta^2}} \right) dy \right]'_v dv, \tag{5}$$

where

$$\varphi(\eta, y; s) = \frac{(\eta - 1)/2 - (\sqrt{1 - 2\eta y + \eta^2})/2 + s}{\eta}$$

is a linear function in s.

Thus, for $j \leq i$ the transition probabilities can be written as

$$P_{ij}(t) = \delta_j^0 + \frac{e^{\mu t/4}}{2\sqrt{2\pi\mu t}} \int_{-\infty}^{\infty} e^{-v^2/(\mu t)} \left[\int_{\cos 2v}^{1} \frac{1}{\sqrt{y - \cos 2v}} \right.$$

$$\left. \times \left(\frac{1}{2\pi\omega} \oint_{0+} C_i^j \left(\frac{\eta - 1 - \sqrt{1 - 2\eta y + \eta^2}}{2\eta} \right)^{i-j} \left(\frac{1}{\eta} \right)^j \frac{d\eta}{\sqrt{1 - 2\eta y + \eta^2}} \right) dy \right]'_v dv.$$

To prove Theorem 2 one should evaluate the integral above and arrive at (4) for the transition probabilities in question. The integral with respect to η is calculated by means of the generating function of Legendre polynomials

$$P_n(y) = \frac{1}{2\pi\omega} \oint_{0+} \frac{1}{\sqrt{1 - 2\eta y + \eta^2}} \frac{d\eta}{\eta^{n+1}}, \quad n = 0, 1, \dots,$$

and gives a linear combination of Legendre polynomials. Having done this, one can use the standard integrals

$$\sin(2k-1)v = \frac{2k-1}{2\sqrt{2}} \int_{\cos 2v}^{1} \frac{P_{k-1}(y)\, dy}{\sqrt{y - \cos 2v}},$$

$$e^{-k(k-1)\mu t} = \frac{e^{\mu t/4}}{\sqrt{\pi \mu t}} \int_{-\infty}^{\infty} e^{-v^2/(\mu t)} \cos(2k-1)v \, dv.$$

In particular, for $j = i, i-1, i-2, \ldots$, by simple calculations we get the following formulas for the transition probabilities:

$$P_{ii}(t) = e^{-i(i-1)\mu t},$$

$$P_{i,i-1}(t) = \frac{i}{2}(e^{-(i-1)(i-2)\mu t} - e^{-i(i-1)\mu t}),$$

$$P_{i,i-2}(t) = \frac{i(i-1)}{4(2i-3)}((i-1)e^{-(i-2)(i-3)\mu t} - (2i-3)e^{-(i-1)(i-2)\mu t} + (i-2)e^{-i(i-1)\mu t}),$$

and so on.

Formula (5) is obtained as the exact close solution of the parabolic equation (3). The method of separation of variables and summation formulas of the special function theory (see [10]) are used.

6. Discussion

Recall an assumption in [4]: the transition probabilities of any pure death process have the following nonlinear property:

$$F_i(t; s) = \mathbf{E}(X_t + sY_t)^i, \quad i = 0, 1, \ldots, \tag{6}$$

where X_t, Y_t are some correlated processes. This assumption is based on the de Finetti–Khinchin theorem on the randomization of an infinite sequence of symmetrically dependent random variables (see [8, Chap. 7, §4]).

Consider a pure death process ξ_t. The process ξ_t stays at the initial state i for a random time τ_i with distribution

$$\mathbf{P}\{\tau_i < t\} = 1 - e^{-\varphi_i t}.$$

At time τ_i the process jumps to the state $i-1$ and evolves further in a similar way.

We consider the initial state i of the process as the existence of i particles of type T (see [2]). The transition of the process to the state $i-1$ is treated as the death of one particle from a collection of i particles. We obtain a random number $\mu^{(l)}(t)$ of descendants for every initial particle l ($l = 1, \ldots, i$; $\mu^{(l)}(t)$ is a random variable taking values 0 or 1). Here, the following relation holds:

$$\xi_t = \mu^{(1)}(t) + \mu^{(2)}(t) + \cdots + \mu^{(i)}(t)$$

(compare with the analogous relation in [1, Chap. 1, §2]). Assume that $\mu^{(l)}(t)$, $l = 1, \ldots, i$, are symmetrically dependent random variables (interchangeable random variables). Then it follows from de Finetti–Khinchin theorem that the branching property for the randomized transition probabilities of the pure death process is expressed by (6) for the generating functions.

7. Concluding Remarks

We call the integral representation (6) the generalized branching property.

In applications [7], the derivation of the generalized branching property is of interest for a polynomial-type process, i.e., when $\mu_i = i^\rho \mu$, $0 < \rho < 1$. This derivation will be given in subsequent papers.

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