THEORY PROBAB. APPL.
Vol. 47, No. 3

Translated from Russian Journal

ABSORPTION PROBABILITY AT THE BORDER OF A RANDOM WALK IN A QUADRANT AND A BRANCHING PROCESS WITH INTERACTION OF PARTICLES*

A. V. KALINKIN†

(*Translated by V. A. Vatutin*)

Abstract. Simple integral representations are obtained for the absorption probability at a boundary point of a random walk on the integer-valued lattice of a quadrant under various hypotheses about the distribution of the jumps of the random walk. To get the representations we apply the method of exponential generating function for solving a stationary first (backward) system of Kolmogorov differential equations suggested in [A. V. Kalinkin, *Theory Probab. Appl.*, 27 (1982), pp. 201–205] and [A. V. Kalinkin, *Sov. Math. Dokl.*, 27 (1983), pp. 493–497].

Key words. absorption probability of a random walk, branching process, exponential generating function, hyperbolic type partial differential equation, Darboux–Picard problem, exact solutions, Chebyshev polynomials

DOI. 10.1137/S0040585X97979810

1. Random walk S_n in a quadrant. Let $\mathbf{N}^2 = \{(\alpha_1, \alpha_2),\ \alpha_1, \alpha_2 = 0, 1, \dots\}$ be the state set of a homogeneous random walk S_n, $n = 0, 1, \dots$. Denote by $P^{(\alpha_1,\alpha_2)}_{(\beta_1,\beta_2)}(n)$ the transition probabilities of the walk in n steps. Assume that the transition probabilities in one step equal

$$\mathbf{P}\big\{S_{n+1} = (\beta_1, \beta_2) \mid S_n = (\alpha_1, \alpha_2)\big\} = p_{\beta_1-\alpha_1+1,\beta_2-\alpha_2+1}$$

if $\alpha_1 > 0, \alpha_2 > 0$, $\beta_1 - \alpha_1 + 1 \geqq 0$, $\beta_2 - \alpha_2 + 1 \geqq 0$;

$$\mathbf{P}\big\{S_{n+1} = (\alpha_1, \alpha_2) \mid S_n = (\alpha_1, \alpha_2)\big\} = 1$$

if $\alpha_1 = 0$ or $\alpha_2 = 0$. Here a probability distribution

$$\left\{p_{\gamma_1\gamma_2} \geqq 0,\ (\gamma_1, \gamma_2) \in \mathbf{N}^2;\ \sum_{\gamma_1,\gamma_2=0}^{\infty} p_{\gamma_1\gamma_2} = 1,\ p_{11} = 0\right\}$$

is specified. Figure 1, case (a) shows possible jumps of the random walk S_n from a state (α_1, α_2). Introduce the generating function

$$h(s_1, s_2) = \sum_{\gamma_1,\gamma_2=0}^{\infty} p_{\gamma_1\gamma_2} s_1^{\gamma_1} s_2^{\gamma_2}.$$

*Received by the editors September 8, 2000.
http://www.siam.org/journals/tvp/47-3/97981.html

†Department of Higher Mathematics, Bauman Moscow State Technical University, 2nd Bauman St. 5, 107005 Moscow, Russia (http://www.bmstu.ru/facult/fn/kalin/kalinkin.htm).

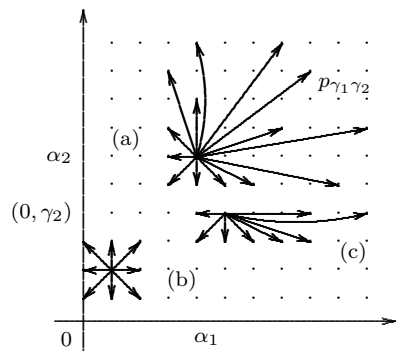

Fig. 1

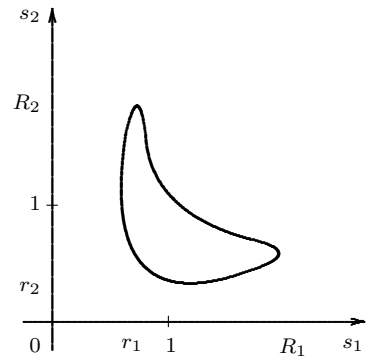

Fig. 2

1.1. Absorption probability problem. The states

$$\{(\gamma_1,0),(0,\gamma_2)\colon\ \gamma_1,\gamma_2\in\mathbf{N}\}$$

are absorbing for S_n. The absorption probabilities are equal:

$$q_{(\gamma_1,0)}^{(\alpha_1,\alpha_2)}=\lim_{n\to\infty}P_{(\gamma_1,0)}^{(\alpha_1,\alpha_2)}(n),\qquad q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)}=\lim_{n\to\infty}P_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)}(n),\tag{1}$$

$(\alpha_1,\alpha_2)\in\mathbf{N}^2$ (by symmetry it suffices to have expressions for probabilities $q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)}$). The problem of finding the absorption probability at point $(0,\gamma_2)$ was investigated by many authors. In [1] the algebraic groups related to random walks in a quadrant are described and investigated, and analytical methods are developed for the Wiener–Hopf equations to solve various problems concerning multidimensional random walks. Report [5] provides an integral representation for $q_{(0,\gamma_2)}^{(1,\alpha_2)}$ in the symmetric case $h(s_1,s_2)=p_{01}s_1^2s_2+p_{10}s_1s_2^2+p_{10}s_1+p_{01}s_2$. If $h(s_1,s_2)=p_{22}s_1^2s_2^2+p_{21}s_1^2s_2+p_{12}s_1s_2^2+p_{20}s_1^2+p_{02}s_2^2+p_{10}s_1+p_{01}s_2+p_{00}$ (Figure 1, case (b)), then the desired expression for the probabilities (1) contains elliptic functions (see [1]). Survey [3] assigns to type 3 (b) the problem of finding the absorption probability at the boundary for a random walk S_n. Malyshev mentioned in [3, p. 21] that "the machinery developed by him can be used to solve partial differential equations with constant coefficients in the quadrant $\mathbf{R}_+^2$."

A differential equation of such form for a problem of type 3 (b) is deduced in section 2. This is partial differential equation (4) with conditions (5) at the boundary of the quadrant $\mathbf{R}_+^2$. Equation (4) specifies the form of integral representation (9) for the probabilities $q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)}$. Theorems 2–4 give, basing on the system of functional equations (16) and (17) deduced in section 3, expressions of form (9) for probabilities (1) under special restrictions on $h(s_1,s_2)$. In a similar way one can derive by (16) and (17) integral representations for the probabilities $q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)}$ in the cases $h(s_1,s_2)=\sum_{k=0}^{\infty}s_1^kh_k(s_1s_2)$, $h(s_1,s_2)=\sum_{k=0}^{\infty}s_2^kh_k(s_1s_2)$, where $h_k(s)$, $k=0,1,\dots$, are positive generating functions. A formula of form (9) for the case $h(s_1,s_2)=p_{20}s_1^2+p_{02}s_2^2+p_{00}$ given in Theorem 5 was deduced by immediately solving (4). The restricted size of this paper does not allow us to give the same derivation (see Remark 3) expressing the probabilities $q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)}$ in the cases $h(s_1,s_2)=p_{20}s_1^2+p_{02}s_2^2+p_{10}s_1+p_{01}s_2+p_{00}$ and $h(s_1,s_2)=p_{22}s_1^2s_2^2+p_{21}s_1^2s_2+p_{12}s_1s_2^2+p_{20}s_1^2+p_{02}s_2^2$. The author failed to get

exact integral representations for probabilities (1) for an arbitrary generating function $h(s_1, s_2)$; the set of particular cases we have considered allows us to make a hypothesis that such an analytic expression of form (9) exists.

In a recent paper [6] Mogulskii and Rogozin applied the factorization method to the general random walk in the quadrant $\mathbf{R}_+^2$. In particular, they found asymptotic formulas for the exit probability of the random walk from the positive quadrant when the exit moment and the location of the exit point tend to infinity.

The exponential representations for the functions $\varphi_1(s)$, $\varphi_2(s)$ we consider follow from the results of [6, Part I] (see expressions for $V_{z+}(\lambda)$, $V_{z-}(\lambda)$, and $v(z, \lambda)$ in [6, Part I]).

2. Branching process with two types of particles and one type of interaction [4]. Consider a time homogeneous Markov process $\mu(t) = (\mu_1(t), \mu_2(t))$ on the phase space $\mathbf{N}^2$ with continuous time t, $t \in [0, \infty)$, and transition probabilities

$$P_{(\beta_1,\beta_2)}^{(\alpha_1,\alpha_2)}(t) = \mathbf{P}\{\mu(t) = (\beta_1, \beta_2) \,|\, \mu(0) = (\alpha_1, \alpha_2)\}.$$

Let, as $\Delta t \to 0$, the transition probabilities have the form ($\lambda > 0$)

$$P_{(\alpha_1,\alpha_2)}^{(\alpha_1,\alpha_2)}(\Delta t) = 1 - \alpha_1\alpha_2\lambda\Delta t + o\,(\Delta t),$$

$$P_{(\beta_1,\beta_2)}^{(\alpha_1,\alpha_2)}(\Delta t) = p_{\beta_1-\alpha_1+1,\beta_2-\alpha_2+1}\alpha_1\alpha_2\lambda\Delta t + o(\Delta t)$$

if $\alpha_1 \neq \beta_1$ or $\alpha_2 \neq \beta_2$. Introduce the generating functions for the transition probabilities ($|s_1| \leqq 1$, $|s_2| \leqq 1$)

$$F_{(\alpha_1,\alpha_2)}(t; s_1, s_2) = \sum_{\beta_1,\beta_2=0}^{\infty} P_{(\beta_1,\beta_2)}^{(\alpha_1,\alpha_2)}(t)\, s_1^{\beta_1} s_2^{\beta_2}, \qquad (\alpha_1, \alpha_2) \in \mathbf{N}^2,$$

$$G_{(\beta_1,\beta_2)}(t; z_1, z_2) = \sum_{\alpha_1,\alpha_2=0}^{\infty} \frac{z_1^{\alpha_1} z_2^{\alpha_2}}{\alpha_1!\,\alpha_2!} P_{(\beta_1,\beta_2)}^{(\alpha_1,\alpha_2)}(t), \qquad (\beta_1, \beta_2) \in \mathbf{N}^2,$$

and the differential operator

$$h\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right) = \sum_{\gamma_1,\gamma_2=0}^{\infty} p_{\gamma_1\gamma_2} \frac{\partial^{\gamma_1+\gamma_2}}{\partial z_1^{\gamma_1}\partial z_2^{\gamma_2}}.$$

The second (direct) system of Kolmogorov differential equations for $\mu(t)$ is equivalent to the linear partial differential equation of the second order (see [4])

$$\frac{\partial F_{(\alpha_1,\alpha_2)}(t; s_1, s_2)}{\partial t} = \lambda(h(s_1, s_2) - s_1 s_2)\, \frac{\partial^2 F_{(\alpha_1,\alpha_2)}(t; s_1, s_2)}{\partial s_1 \partial s_2}$$

with the initial condition $F_{(\alpha_1,\alpha_2)}(0; s_1, s_2) = s_1^{\alpha_1} s_2^{\alpha_2}$. The first (backward) system of Kolmogorov differential equations for $\mu(t)$ is equivalent to the linear equation (see [4]; compare with [18, Theorem 1])

$$\frac{\partial G_{(\beta_1,\beta_2)}(t; z_1, z_2)}{\partial t} = \lambda z_1 z_2 \left(h\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right) - \frac{\partial^2}{\partial z_1 \partial z_2} \right) G_{(\beta_1,\beta_2)}(t; z_1, z_2)$$

with the initial condition $G_{(\beta_1,\beta_2)}(0; z_1, z_2) = z_1^{\beta_1} z_2^{\beta_2}/(\beta_1!\,\beta_2!)$.

The event $\{\mu(t) = (\beta_1, \beta_2)\}$ for this branching process is treated as the existence at moment t of β_1 particles of type T_1 and β_2 particles of type T_2 (male and female individuals [4], [8]). Let vector (α_1, α_2) be the initial state of the process. One may

assume that the interaction of a type T_1 particle with a type T_2 particle occurs after a random time $\tau_{(\alpha_1,\alpha_2)}$, $\mathbf{P}\{\tau_{(\alpha_1,\alpha_2)} < t\} = 1 - e^{-\alpha_1\alpha_2\lambda t}$. This pair of particles transforms, independently of the other particles, into a new group consisting of γ_1 particles of type T_1 and γ_2 particles of type T_2: $T_1 + T_2 \to \gamma_1 T_1 + \gamma_2 T_2$, in accordance with the probability distribution $\{p_{\gamma_1\gamma_2}\}$. As a result the process passes to the state corresponding to the vector $(\alpha_1 + \gamma_1 - 1,\ \alpha_2 + \gamma_2 - 1)$. The subsequent evolution of the stochastic process proceeds in a similar way. The branching process $\mu(t)$ may degenerate in one of the absorbing states belonging to the set $\{(\gamma_1, 0), (0, \gamma_2)\colon\ \gamma_1, \gamma_2 \in \mathbf{N}\}$ (particles of only one type remain and, therefore, interactions $T_1 + T_2$ are impossible).

2.1. Stationary first equation for the exponential generating function. It follows from the description of $\mu(t)$ that observing $\mu(t)$ only at the moment of changing its states we get an "embedded Markov chain," which coincides with the random walk S_n we have introduced in section 1. The extinction probability of $\mu(t)$ coincides with the absorption probability of S_n under one and the same initial condition (compare with [2, Chap. 2, section 1, Theorem 1]):

$$q_{(\gamma_1,0)}^{(\alpha_1,\alpha_2)} = \lim_{t\to\infty} P_{(\gamma_1,0)}^{(\alpha_1,\alpha_2)}(t), \quad q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)} = \lim_{t\to\infty} P_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)}(t), \qquad (\alpha_1,\alpha_2) \in \mathbf{N}^2.$$

We introduce the exponential generating function for $q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)}$:

$$g_{(0,\gamma_2)}(z_1, z_2) = \sum_{\alpha_1,\alpha_2=0}^{\infty} \frac{z_1^{\alpha_1} z_2^{\alpha_2}}{\alpha_1!\,\alpha_2!} q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)}. \tag{2}$$

Clearly, $g_{(0,\gamma_2)}(z_1, z_2)$ is an analytic function in variables z_1 and z_2, since

$$\left|g_{(0,\gamma_2)}(z_1, z_2)\right| \leqq \sum_{\alpha_1,\alpha_2=0}^{\infty} \frac{|z_1|^{\alpha_1}\,|z_2|^{\alpha_2}}{\alpha_1!\,\alpha_2!} \left|q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)}\right| \leqq e^{|z_1|+|z_2|}. \tag{3}$$

Similarly to Theorem 2 in [18] one can show that

$$g_{(0,\gamma_2)}(z_1, z_2) = \lim_{t\to\infty} G_{(0,\gamma_2)}(t; z_1, z_2)$$

and that $g_{(0,\gamma_2)}(z_1, z_2)$ satisfies the stationary first equation

$$\left[h\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right) - \frac{\partial^2}{\partial z_1 \partial z_2}\right] g_{(0,\gamma_2)}(z_1, z_2) = 0. \tag{4}$$

Consider the absorbing states $(0, \gamma_2)$, $\gamma_2 = 1, 2, \ldots$. By means of the evident equalities for absorption probabilities $q_{(0,\gamma_2)}^{(0,\gamma_2)} = 1$, $q_{(0,\gamma_2)}^{(0,\alpha_2)} = 0$ if $\alpha_2 \neq \gamma_2$; $q_{(0,\gamma_2)}^{(\alpha_1,0)} = 0$ if $\alpha_1 = 0, 1, \ldots$, we deduce the needed conditions at the boundaries $z_1 = 0$ and $z_2 = 0$ of $\mathbf{R}_+^2$:

$$g_{(0,\gamma_2)}(0, z_2) = \frac{z_2^{\gamma_2}}{\gamma_2!}, \quad g_{(0,\gamma_2)}(z_1, 0) = 0. \tag{5}$$

The respective boundary conditions $g_{(0,0)}(0, z_2) = 1$, $g_{(0,0)}(z_1, 0) = 1$ for the absorbing state $(0, 0)$ can be found similarly to the derivation of (5).

2.2. On the integral representation for the probabilities $q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)}$. One can find in survey [3] a list of papers dealing with the boundary problem of type (4), (5). Considering problem (4), (5), i.e., the problem of finding an explicit expression for $g_{(0,\gamma_2)}(z_1,z_2)$, one can apply some approaches borrowed from the general theory of linear partial differential operators with constant coefficients [11]. The characteristic equation for (4) is

$$h(s_1,s_2)-s_1s_2=0 \tag{6}$$

(cf. [2, Chap. 2, section 1, equation (9)]). Let $\varphi_1(s_2)$, $\varphi_2(s_1)$ be the functions specified by the characteristic equation $h(\varphi_1(s_2),s_2)-\varphi_1(s_2)\,s_2=0$; $h(s_1,\varphi_2(s_1))-s_1\varphi_2(s_1)=0$. The "general solution" of (4) has the form (see [11, section 2] for the exponential representation of the solutions of partial differential equations)

$$\int_Q e^{z_1\varphi_1(s_2)+z_2s_2}\,\mu^1(ds_2)=\int_Q e^{z_1s_1+z_2\varphi_2(s_1)}\,\mu^2(ds_1), \tag{7}$$

where $\mu^1(ds_2)$ and $\mu^2(ds_1)$ are measures concentrated on $Q=\{(s_1,s_2)\colon\ h(s_1,s_2)-s_1s_2=0\}$. Indeed insert, for instance, the first of the expressions in (7) into (4). After formal calculations we obtain

$$\begin{aligned}&\left[h\left(\frac{\partial}{\partial z_1},\ \frac{\partial}{\partial z_2}\right)-\frac{\partial^2}{\partial z_1\partial z_2}\right]\int_Q e^{z_1\varphi_1(s_2)+z_2s_2}\,\mu^1(ds_2)\\&\quad=\int_Q\left[h\left(\frac{\partial}{\partial z_1},\ \frac{\partial}{\partial z_2}\right)-\frac{\partial^2}{\partial z_1\partial z_2}\right]e^{z_1\varphi_1(s_2)+z_2s_2}\,\mu^1(ds_2)\\&\quad=\int_Q e^{z_1\varphi_1(s_2)+z_2s_2}\big(h(\varphi_1(s_2),s_2)-\varphi_1(s_2)\,s_2\big)\,\mu^1(ds_2)=0.\end{aligned}$$

By (7), the exponent expansion

$$e^{z_1s_1+z_2s_2}=\sum_{\alpha_1,\alpha_2=0}^{\infty}\frac{z_1^{\alpha_1}z_2^{\alpha_2}}{\alpha_1!\,\alpha_2!}\,s_1^{\alpha_1}s_2^{\alpha_2}, \tag{8}$$

definition (2) of generating function $g_{(0,\gamma_2)}(z_1,z_2)$, and taking into account that the measures in (7) depend on boundary conditions (5), we conclude that the desired expression for the absorption probability at point $(0,\gamma_2)$ has the form

$$q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)}=\int_Q\varphi_1^{\alpha_1}(s_2)\,s_2^{\alpha_2}\,\mu^1_{(0,\gamma_2)}(ds_2)=\int_Q s_1^{\alpha_1}\varphi_2^{\alpha_2}(s_1)\,\mu^2_{(0,\gamma_2)}(ds_1). \tag{9}$$

3. Functional equation system for the generating function of absorption probabilities. Introduce the generating functions

$$\Phi_1^{(\alpha_1,\alpha_2)}(s)=\sum_{\gamma_1=0}^{\infty}q_{(\gamma_1,0)}^{(\alpha_1,\alpha_2)}s^{\gamma_1},\quad \Phi_2^{(\alpha_1,\alpha_2)}(s)=\sum_{\gamma_2=0}^{\infty}q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)}s^{\gamma_2},$$

$(\alpha_1,\alpha_2)\in\mathbf{N}^2$ (analytic within the circle $|s|\leqq 1$). We need auxiliary notation to deduce the needed system of equations for $\Phi_1^{(\alpha_1,\alpha_2)}(s)$, $\Phi_2^{(\alpha_1,\alpha_2)}(s)$.

3.1. Random walk $\widehat{S}_n$ in the right half-plane. Random walk $\widetilde{S}_n$ in the upper half-plane. Introduce a random walk $\widehat{S}_n$, $n = 0, 1, \dots$, on the set $\mathbf{N} \times \mathbf{Z} = \{(\alpha_1, \alpha_2)\colon\ \alpha_1 \in \mathbf{N},\ \alpha_2 \in \mathbf{Z}\}$, where $\mathbf{Z} = \{\dots, -1, 0, 1, \dots\}$. By $\widehat{P}^{(\alpha_1,\alpha_2)}_{(\beta_1,\beta_2)}(n)$ we denote the transition probabilities in n steps. The transition probabilities in one step are assumed to be equal to $\mathbf{P}\{\widehat{S}_{n+1} = (\beta_1, \beta_2) \,|\, \widehat{S}_n = (\alpha_1, \alpha_2)\} = p_{\beta_1-\alpha_1+1,\beta_2-\alpha_2+1}$ if $\alpha_1 > 0$ and $\beta_1 - \alpha_1 + 1 \geqq 0$, $\beta_2 - \alpha_2 + 1 \geqq 0$; $\mathbf{P}\{\widehat{S}_{n+1} = (\alpha_1, \alpha_2) \,|\, \widehat{S}_n = (\alpha_1, \alpha_2)\} = 1$ if $\alpha_1 = 0$. The states $\{(0, \gamma_2)\colon\ \gamma_2 \in \mathbf{Z}\}$ are absorbing for $\widehat{S}_n$. Clearly, the limits $\widehat{q}^{(\alpha_1,\alpha_2)}_{(0,\gamma_2)} = \lim_{n\to\infty} \widehat{P}^{(\alpha_1,\alpha_2)}_{(0,\gamma_2)}(n)$, $(\alpha_1, \alpha_2) \in \mathbf{N} \times \mathbf{Z}$, $\gamma_2 \in \mathbf{Z}$ exist (the absorption probabilities at the boundary points $(0, \gamma_2)$).

In a similar way we define a random walk $\widetilde{S}_n$, $n = 0, 1, \dots$, on the set $\mathbf{Z} \times \mathbf{N} = \{(\alpha_1, \alpha_2)\colon\ \alpha_1 \in \mathbf{Z},\ \alpha_2 \in \mathbf{N}\}$. By $\widetilde{P}^{(\alpha_1,\alpha_2)}_{(\beta_1,\beta_2)}(n)$ we denote the transition probabilities in n steps. The transition probabilities in one step are assumed to be equal to $\mathbf{P}\{\widetilde{S}_{n+1} = (\beta_1, \beta_2) \,|\, \widetilde{S}_n = (\alpha_1, \alpha_2)\} = p_{\beta_1-\alpha_1+1,\beta_2-\alpha_2+1}$ if $\alpha_2 > 0$ and $\beta_1 - \alpha_1 + 1 \geqq 0$, $\beta_2 - \alpha_2 + 1 \geqq 0$; $\mathbf{P}\{\widetilde{S}_{n+1} = (\alpha_1, \alpha_2) \,|\, \widetilde{S}_n = (\alpha_1, \alpha_2)\} = 1$ if $\alpha_2 = 0$. The states $\{(\gamma_1, 0)\colon\ \gamma_1 \in \mathbf{Z}\}$ are absorbing for $\widetilde{S}_n$. The limits $\widetilde{q}^{(\alpha_1,\alpha_2)}_{(\gamma_1,0)} = \lim_{n\to\infty} \widetilde{P}^{(\alpha_1,\alpha_2)}_{(\gamma_1,0)}(n)$, $(\alpha_1, \alpha_2) \in \mathbf{Z} \times \mathbf{N}$, $\gamma_1 \in \mathbf{Z}$ exist. Introduce the functions

$$\varphi_1^{(\alpha_1,\alpha_2)}(s) = \sum_{\gamma_2=-\infty}^{\infty} \widehat{q}^{(\alpha_1,\alpha_2)}_{(0,\gamma_2)} s^{\gamma_2},$$

$$(\alpha_1, \alpha_2) \in \mathbf{N} \times \mathbf{Z}, \quad \varphi_1(s) = \varphi_1^{(1,0)}(s); \tag{10}$$

$$\varphi_2^{(\alpha_1,\alpha_2)}(s) = \sum_{\gamma_1=-\infty}^{\infty} \widetilde{q}^{(\alpha_1,\alpha_2)}_{(\gamma_1,0)} s^{\gamma_1},$$

$$(\alpha_1, \alpha_2) \in \mathbf{Z} \times \mathbf{N}, \quad \varphi_2(s) = \varphi_2^{(0,1)}(s). \tag{11}$$

Series (10) and (11) are Laurent series [9]. Let $\varphi_1(s)$ be convergent in a ring $r_2 < |s| < R_2$ and let $\varphi_2(s)$ be convergent in a ring $r_1 < |s| < R_1$. Note that $r_i \leqq 1 \leqq R_i$, $i = 1, 2$.

3.2. Deduction of a system of equations. Further, for simplicity, we restrict ourselves by the case when $h(s_1, s_2)$ is an analytic function in its arguments for all s_1, s_2.

LEMMA 1. *The following relations are valid for the generating functions in question*:

$$\varphi_1^{(\alpha_1,\alpha_2)}(s) = \varphi_1^{\alpha_1}(s)\, s^{\alpha_2}, \qquad (\alpha_1, \alpha_2) \in \mathbf{N} \times \mathbf{Z}, \tag{12}$$

$$h(\varphi_1(s), s) - \varphi_1(s)\, s = 0, \qquad r_2 < |s| < R_2, \tag{13}$$

$$\varphi_2^{(\alpha_1,\alpha_2)}(s) = s^{\alpha_1} \varphi_2^{\alpha_2}(s), \qquad (\alpha_1, \alpha_2) \in \mathbf{Z} \times \mathbf{N}, \tag{14}$$

$$h(s, \varphi_2(s)) - s\varphi_2(s) = 0, \qquad r_1 < |s| < R_1. \tag{15}$$

Proof. We derive relations (12) and (13) only; relations (14) and (15) can be deduced in a similar way. The absorption probabilities of $\widehat{S}_n$ satisfy the equality $\widehat{q}^{(\alpha_1,\alpha_2)}_{(0,\gamma_2)} = \widehat{q}^{(\alpha_1,0)}_{(0,\gamma_2-\alpha_2)}$, $(\alpha_1, \alpha_2) \in \mathbf{N} \times \mathbf{Z}$, $\gamma_2 \in \mathbf{Z}$. Hence, on account of definition (10) we obtain (here and in what follows, $r_2 < |s| < R_2$) $\varphi_1^{(\alpha_1,\alpha_2)}(s) = \varphi_1^{(\alpha_1,0)}(s)\, s^{\alpha_2}$, $(\alpha_1, \alpha_2) \in \mathbf{N} \times \mathbf{Z}$. By the strong Markov property of the process $\widehat{S}_n$, $n = 0, 1, \dots$,

we have $\widehat{q}_{(0,\gamma_2)}^{(\alpha_1,0)} = \sum_{l=-\infty}^{\infty} \widehat{q}_{(0,l)}^{(1,0)} \widehat{q}_{(0,\gamma_2)}^{(\alpha_1-1,l)}$, $\alpha_1 = 1,2,\dots$, $\gamma_2 \in \mathbf{Z}$. Convolution by means of function (10) gives (the permutation of the order of summation is correct) $\varphi_1^{(\alpha_1,0)}(s) = \varphi_1(s)\,\varphi_1^{(\alpha_1-1,0)}(s)$, $\alpha_1 = 1,2,\dots$, and, therefore,

$$\varphi_1^{(\alpha_1,0)}(s) = \varphi_1^{\alpha_1}(s), \qquad \alpha_1 \in \mathbf{N}.$$

Relation (12) is established. Relation (13) also follows from the strong Markov property of $\widehat{S}_n$:

$$\begin{aligned}\varphi_1(s)\,s = \varphi_1^{(1,1)}(s) &= \sum_{\alpha_1,\alpha_2=0}^{\infty} p_{\alpha_1\alpha_2}\varphi_1^{(\alpha_1,\alpha_2)}(s)\\ &= \sum_{\alpha_1,\alpha_2=0}^{\infty} p_{\alpha_1\alpha_2}\varphi_1^{\alpha_1}(s)\,s^{\alpha_2} = h(\varphi_1(s),s).\end{aligned}$$

Lemma 1 is proved.

THEOREM 1. *The following relations are valid for the generating functions of absorption probabilities*:

$$\Phi_1^{(\alpha_1,\alpha_2)}(s) = s^{\alpha_1}\varphi_2^{\alpha_2}(s) - \Phi_2^{(\alpha_1,\alpha_2)}(\varphi_2(s)) + \Phi_2^{(\alpha_1,\alpha_2)}(0), \qquad r_1 < |s| < R_1; \tag{16}$$

$$\Phi_2^{(\alpha_1,\alpha_2)}(s) = \varphi_1^{\alpha_1}(s)\,s^{\alpha_2} - \Phi_1^{(\alpha_1,\alpha_2)}(\varphi_1(s)) + \Phi_1^{(\alpha_1,\alpha_2)}(0), \qquad r_2 < |s| < R_2. \tag{17}$$

Proof. We demonstrate (17) only; relation (16) can be deduced in a similar way. Taking into account the strong Markov properties of S_n and $\widehat{S}_n$ we have

$$q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)} = \widehat{q}_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)} - \sum_{\gamma_1=1}^{\infty} q_{(\gamma_1,0)}^{(\alpha_1,\alpha_2)}\widehat{q}_{(0,\gamma_2)}^{(\gamma_1,0)}, \qquad (\alpha_1,\alpha_2)\in\mathbf{N}^2, \quad \gamma_2\in\mathbf{N};$$

$$\widehat{q}_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)} = \sum_{\gamma_1=1}^{\infty} q_{(\gamma_1,0)}^{(\alpha_1,\alpha_2)}\widehat{q}_{(0,\gamma_2)}^{(\gamma_1,0)}, \qquad (\alpha_1,\alpha_2)\in\mathbf{N}^2, \quad \gamma_2 = -1,-2,\dots.$$

By means of the first equality we get (for $|s| < R_2$ the series in question are absolutely convergent and, therefore, the permutations of the order of summation are correct):

$$\sum_{\gamma_2=0}^{\infty} q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)} s^{\gamma_2} = \sum_{\gamma_2=0}^{\infty} \widehat{q}_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)} s^{\gamma_2} - \sum_{\gamma_1=1}^{\infty} q_{(\gamma_1,0)}^{(\alpha_1,\alpha_2)} \sum_{\gamma_2=0}^{\infty} \widehat{q}_{(0,\gamma_2)}^{(\gamma_1,0)} s^{\gamma_2}. \tag{18}$$

Further, by the second equality we conclude

$$0 = \sum_{\gamma_2=-1}^{-\infty} \widehat{q}_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)} s^{\gamma_2} - \sum_{\gamma_1=1}^{\infty} q_{(\gamma_1,0)}^{(\alpha_1,\alpha_2)} \sum_{\gamma_2=-1}^{-\infty} \widehat{q}_{(0,\gamma_2)}^{(\gamma_1,0)} s^{\gamma_2}, \qquad |s| > r_2.$$

Summing the obtained expressions and taking into account (12) we see that

$$\begin{aligned}\sum_{\gamma_2=0}^{\infty} q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)} s^{\gamma_2} &= \varphi_1^{(\alpha_1,\alpha_2)}(s) - \sum_{\gamma_1=1}^{\infty} q_{(\gamma_1,0)}^{(\alpha_1,\alpha_2)}\varphi_1^{(\gamma_1,0)}(s)\\ &= \varphi_1^{\alpha_1}(s)\,s^{\alpha_2} - \sum_{\gamma_1=0}^{\infty} q_{(\gamma_1,0)}^{(\alpha_1,\alpha_2)}\varphi_1^{\gamma_1}(s) + q_{(0,0)}^{(\alpha_1,\alpha_2)}, \qquad r_2 < |s| < R_2.\end{aligned}$$

The last coincides with (17). Theorem 1 is proved.

Equality (18) shows that $\Phi_i^{(\alpha_1,\alpha_2)}(s)$ are analytic functions in the circle $|s| < R_i$ ($i = 1,2$; $(\alpha_1,\alpha_2)\in\mathbf{N}^2$).

4. Analysis of the characteristic equation $h(s_1, s_2) - s_1 s_2 = 0$. Consider the functions $\varphi_1(s)$, $\varphi_2(s)$ defined in relations (10) and (11) and the set $Q^+ = \{(s_1, s_2)\colon\ h(s_1, s_2) - s_1 s_2 = 0,\ s_1 > 0,\ s_2 > 0\}$.

Denote by S the set of all points in the plane with integer-valued coordinates (γ_1, γ_2) for which either $p_{\gamma_1+1,\gamma_2+1} > 0$ or $\gamma_1 = \gamma_2 = 0$. Let S_L be the lattice of integer-valued points in the plane which contains S and includes no sublattice possessing this property. The coordinates of all the points of lattice S_L can be obtained by arranging all possible linear combinations with integer coefficients from the coordinates of the points of S (see [2, Chap. 5, section 4]). By S_L^+ we denote the set of all points which can be obtained by arranging all possible linear combinations with integer nonnegative coefficients from the coordinates of the points of set S. We say that a probability distribution $\{p_{\gamma_1\gamma_2}, (\gamma_1, \gamma_2) \in \mathbf{N}^2\}$ meets *condition* A if S_L coincides with S_L^+ and S_L is a two-dimensional lattice. Under condition A lattice S_L describes the set of states which are accessible for stochastic process $\mu(t)$ from the initial state (α_1, α_2): if $(\beta_1, \beta_2) \in S_L$ and $\alpha_1+\beta_1+1 \geqq 0$, $\alpha_2+\beta_2+1 \geqq 0$, then $P^{(\alpha_1,\alpha_2)}_{(\alpha_1+\beta_1+1,\alpha_2+\beta_2+1)}(t_0) > 0$ for some $t_0 < \infty$; if $(\beta_1, \beta_2) \notin S_L$, then $P^{(\alpha_1,\alpha_2)}_{(\alpha_1+\beta_1+1,\alpha_2+\beta_2+1)}(t) \equiv 0$.

LEMMA 2. *Let $\{p_{\gamma_1\gamma_2}, (\gamma_1, \gamma_2) \in \mathbf{N}^2\}$ be a probability distribution meeting condition* A. *Then* $0 < r_2 \leqq 1 \leqq R_2 < \infty$, $0 < \varphi_1(r_2) < \infty$, $0 < \varphi_1(R_2) < \infty$; $0 < r_1 \leqq 1 \leqq R_1 < \infty$, $0 < \varphi_2(r_1) < \infty$, $0 < \varphi_2(R_1) < \infty$.

Proof. We demonstrate the statement of the lemma for $\varphi_1(s)$ only; analysis of $\varphi_2(s)$ can be performed in a similar way. Condition A implies that expansion (10) for $\varphi_1(s)$ contains the principal and regular parts [9]. It follows from (10) that $\varphi_1(s) > 0$ for $r_2 < s < R_2$.

Consider r_2 and $\varphi_1(r_2)$.

A) Let $p_{\gamma_1 0} = 0$ for all $\gamma_1 \geqq 1$. By condition A, $p_{00} > 0$ and there exists a point (β_1, β_2) such that $0 < \beta_2 < \beta_1$, $p_{\beta_1\beta_2} > 0$. Equality (13) implies $\varphi_1(s)\, s \geqq p_{\beta_1\beta_2}\varphi_1^{\beta_1}(s)\, s^{\beta_2}$ or $1 \geqq p_{\beta_1\beta_2}\varphi_1^{\beta_1-1}(s)\, s^{\beta_2-1}$ (here and in what follows, we deal with the interval $r_2 < s < R_2$). If $\beta_2 = 1$, then $\varphi_1(s) \leqq p_{\beta_1\beta_2}^{1/(\beta_1-1)}$ $(\beta_1 > 1)$ and letting $s \downarrow r_2$ in the last inequality we see that $\varphi_1(r_2)$ is finite. If $r_2 = 0$, then there is no principal part in expansion (10) for $\varphi_1(s)$ contradicting condition A. If $\beta_2 > 1$, we likewise use the inequality $1 \geqq p_{\beta_1\beta_2}\varphi_1^{\beta_1-\beta_2}(s)\,(\varphi_1(s)\, s)^{\beta_2-1} \geqq p_{\beta_1\beta_2}\varphi_1^{\beta_1-\beta_2}(s) p_{00}^{\beta_2-1}$, where $\beta_1 - \beta_2 > 0$.

B) Let there exist a point $(\gamma_1, 0)$ such that $\gamma_1 \geqq 1$ and $p_{\gamma_1 0} > 0$. In this case (13) implies $s \geqq p_{\gamma_1 0}\varphi_1^{\gamma_1-1}(s)$. If $\gamma_1 > 1$, the last inequality yields $\varphi_1(r_2) < \infty$ implying $r_2 > 0$. If $\gamma_1 = 1$, then $s \geqq p_{\gamma_1 0}$ and $r_2 > 0$. Under condition A there exists a point (β_1, β_2) such that $\beta_1 > 1$, $p_{\beta_1\beta_2} > 0$; by means of (13) we obtain $s \geqq p_{\beta_1\beta_2}\varphi_1^{\beta_1-1}(s)\, s^{\beta_2}$ and, letting $s \downarrow r_2$ conclude that $\varphi_1(r_2)$ is finite.

Consider R_2 and $\varphi_1(R_2)$.

A) Let there exist a point (γ_1, γ_2) such that $\gamma_1 \geqq 1$, $\gamma_2 \geqq 1$, and $p_{\gamma_1\gamma_2} > 0$ (note that either $\gamma_1 > 1$ or $\gamma_2 > 1$ in view of $p_{11} = 0$); equation (13) gives $1 \geqq p_{\gamma_1\gamma_2}\varphi_1^{\gamma_1-1}(s)\, s^{\gamma_2-1}$. If $\gamma_1 > 1$ and $\gamma_2 = 1$, then $1 \geqq p_{\gamma_1\gamma_2}\varphi_1^{\gamma_1-1}(s)$. Assuming $R_2 = \infty$ we obtain $\limsup_{s\to\infty} \varphi_1(s) < \infty$. Hence, provided the assumption is true, expansion (10) of $\varphi_1(s)$ has no principal part contradicting condition A. If $\gamma_1 = 1$ and $\gamma_2 > 1$, then $1 \geqq p_{\gamma_1\gamma_2} s^{\gamma_2-1}$. Hence $R_2 < \infty$. Condition A provides existence of a point (β_1, β_2) such that $\beta_1 > 1$ and $p_{\beta_1\beta_2} > 0$. Letting $s \uparrow R_2$ in the inequality $s \geqq p_{\beta_1\beta_2}\varphi_1^{\beta_1-1}(s)\, s^{\beta_2}$, we see that $\varphi_1(R_2)$ is finite. If $\gamma_1 > 1$ and $\gamma_2 > 1$, then similar arguments lead us to the same conclusion.

B) Let $p_{\gamma_1\gamma_2} = 0$ for all $\gamma_1 \geqq 1$ and $\gamma_2 \geqq 1$. By condition A there are points $(\gamma_1, 0)$

and $(0,\gamma_2)$ such that $\gamma_1 \geqq 2$, $p_{\gamma_1 0} > 0$ and $\gamma_2 \geqq 2$, $p_{0\gamma_2} > 0$. Moreover, either $\gamma_1 > 2$ or $\gamma_2 > 2$; by (13) we obtain $s \geqq p_{\gamma_1 0}\varphi_1^{\gamma_1-1}(s)$ and, respectively, $\varphi_1(s) \geqq p_{0\gamma_2}s^{\gamma_2-1}$. If $\gamma_1 > 2$, then $s \geqq p_{\gamma_1 0}(p_{0\gamma_2}s^{\gamma_2-1})^{\gamma_1-1}$. Hence, in view of $(\gamma_1-1)(\gamma_2-1) > 1$, we see that $R_2 < \infty$, implying $\varphi_1(R_2) < \infty$. If $\gamma_2 > 2$, then $\varphi_1(s) \geqq p_{0\gamma_2}(p_{\gamma_1 0}\varphi_1^{\gamma_1-1}(s))^{\gamma_2-1}$ and, further, one should proceed in a similar way. Lemma 2 is proved.

Since expansion (10) for $\varphi_1(s)$ has nonnegative coefficients, $\varphi_1(s)$ is convex in (r_2, R_2) while the points $s = r_2$ and $s = R_2$ are singular for $\varphi_1(s)$. Following [2, Chap. 5, section 4] one can show that under condition A these points are branch points of order two. If $h(s_1, s_2)$ is a polynomial, then it follows from the Pellet theorem [18] that for any C the set Q^+ and the line $s_1 = C$ (or the line $s_2 = C$) have at most two common points. Therefore, $r_1 < \varphi_1(r_2) < R_1$, $r_1 < \varphi_1(R_2) < R_1$ and $r_2 < \varphi_2(r_1) < R_2$, $r_2 < \varphi_2(R_1) < R_2$ in this case. One can see in Figure 2 the set Q^+ provided the conditions above are valid. To investigate properties of Q^+ in more detail it is necessary to deal with the *criticality parameters* [2]

$$a_1 = \frac{\partial h(s_1,s_2)}{\partial s_1}\bigg|_{s_1=1,s_2=1} - 1, \quad a_2 = \frac{\partial h(s_1,s_2)}{\partial s_2}\bigg|_{s_1=1,s_2=1} - 1.$$

5. Integral representation for $q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)}$ when $h(s_1,s_2) = s_2h_1(s_1,s_2)$. Put $h(s_1,s_2) = s_2h_1(s_1,s_2)$, where $h_1(s_1,s_2)$ is a probability generating function.

LEMMA 3 (see [2]). *Let $h(s_1,s_2) = s_2h_1(s_1,s_2)$ and the distribution $\{p_{\gamma_1\gamma_2}, (\gamma_1,\gamma_2) \in \mathbf{N}^2\}$ be such that there are probabilities $p_{0\gamma_2} > 0$, $p_{\gamma_1\gamma_2} > 0$ with $\gamma_1 \geqq 2$ and $p_{\gamma_1\gamma_2} > 0$ with $\gamma_2 > 1$. Then $\varphi_1(s)$ is an analytic function in the circle $|s| < R_2$, $1 \leqq R_2 < \infty$, and $0 < \varphi_1(R_2) < \infty$.*

Under the conditions of Lemma 3 the lattice S_L is two-dimensional.

THEOREM 2 (see [2]). *Let $h(s_1,s_2) = s_2h_1(s_1,s_2)$ and the conditions of Lemma 3 be valid. Then the absorption probability equals* ($\alpha_1 \in \mathbf{N}$, $\alpha_2 = 1,2,\dots$)

$$q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)} = \frac{1}{2\pi i}\oint_{0+} \varphi_1^{\alpha_1}(s)\, s^{\alpha_2}\, \frac{ds}{s^{\gamma_2+1}}, \qquad \gamma_2 \in \mathbf{N}. \tag{19}$$

Proof. From the definition of $\Phi_2^{(\alpha_1,\alpha_2)}(s)$ it follows that the absorption probability is expressed by the integral

$$q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)} = \frac{1}{2\pi i}\int_{\theta_\rho^+} \Phi_2^{(\alpha_1,\alpha_2)}(s)\, \frac{ds}{s^{\gamma_2+1}}, \qquad (\alpha_1,\alpha_2) \in \mathbf{N}^2, \quad \gamma_2 \in \mathbf{N}, \tag{20}$$

where the integration contour is a circle $\theta_\rho\colon |s| = \rho$, $0 < \rho < R_2$. Under the condition $h(s_1,s_2) = s_2h_1(s_1,s_2)$ on the jumps of the random walk S_n we have $q_{(\gamma_1,0)}^{(\alpha_1,\alpha_2)} = 0$. Therefore, $\Phi_1^{(\alpha_1,\alpha_2)}(s) \equiv 0$ ($\alpha_2 = 1,2,\dots$) and by (17) we obtain $\Phi_2^{(\alpha_1,\alpha_2)}(s) = \varphi_1^{\alpha_1}(s)\, s^{\alpha_2}$. Substituting the last expression in (20) and taking into account that $\varphi_1(s)$ is analytic in the circle $|s| < R_2$ we get (19). Theorem 2 is proved.

Remark 1. Equation (4) has the form

$$\left[h_1\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right) - \frac{\partial}{\partial z_1}\right] g_{(0,\gamma_2)}(z_1,z_2) = 0. \tag{21}$$

The characteristic equation $h_1(s_1,s_2) - s_1 = 0$ and the function $\varphi_1(s)$ are investigated in [2, Chap. 5, section 4]. Equation (21) coincides with the stationary first Kolmogorov equation for a branching process (with independent particles) with one

final type T_2 and one nonfinal type T_1 of particles (see [2]). A direct solution of (21) is contained in [19] (the case $k = 1$).

6. Integral representation for $q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)}$ in the case $h(s_1, s_2) = h_0(s_1) + s_2 h_1(s_1)$. Set $h(s_1, s_2) = h_0(s_1) + s_2 h_1(s_1)$, where $h_0(s_1)$, $h_1(s_1)$ are positive generating functions. Figure 1, case (c) shows possible jumps of S_n.

LEMMA 4. *Let $h(s_1, s_2) = h_0(s_1) + s_2 h_1(s_1)$ and, for the distribution $\{p_{\gamma_1\gamma_2}, (\gamma_1, \gamma_2) \in \mathbf{N}^2\}$, let there be probabilities $p_{0\gamma_2} > 0$, $p_{\gamma_1\gamma_2} > 0$ with $\gamma_1 \geqq 2$ and $p_{\gamma_1 0} > 0$. Then $\varphi_1(s)$ is an analytic function in the domain $|s| > r_2$, $0 < r_2 \leqq 1$, and the value $\varphi_1(r_2)$ belongs to the interval (r_1, R_1), $0 \leqq r_1 < 1 < R_1 \leqq \infty$.*

Under the conditions of Lemma 4 S_L is a two-dimensional lattice. The proof of Lemma 4 is analogous to those of Lemmas 2 and 3. To this aim it is necessary to use the explicit expression $\varphi_2(s) = h_0(s)/(s - h_1(s))$ which follows from the equation $h_0(s_1) + s_2 h_1(s_1) - s_1 s_2 = 0$.

THEOREM 3. *Let $h(s_1, s_2) = h_0(s_1) + s_2 h_1(s_1)$ and the conditions of Lemma 4 be valid. The absorption probability equals ($\alpha_1 \in \mathbf{N}$, $\alpha_2 = 1, 2, \dots$)*

$$q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)} = \frac{1}{2\pi i} \int_{\theta_\rho^+} \varphi_1^{\alpha_1}(s)\, s^{\alpha_2} \frac{ds}{s^{\gamma_2+1}}, \qquad \gamma_2 = 1, 2, \dots,$$

where the integration contour is a circle θ_ρ: $|s| = \rho$, $\rho > r_2$.

Proof. We use the integral representation (20) as a starting point. Substituting (17) for $\Phi_2^{(\alpha_1,\alpha_2)}(s)$ we obtain ($\gamma_2 = 1, 2, \dots$)

$$q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)} = \frac{1}{2\pi i} \int_{\theta_\rho^+} \varphi_1^{\alpha_1}(s)\, s^{\alpha_2} \frac{ds}{s^{\gamma_2+1}} - \frac{1}{2\pi i} \int_{\theta_\rho^+} \Phi_1^{(\alpha_1,\alpha_2)}\big(\varphi_1(s)\big) \frac{ds}{s^{\gamma_2+1}},$$

where $\rho > r_2$. The second integral equals zero, since the condition $h(s_1, s_2) = h_0(s_1) + s_2 h_1(s_1)$ on the jumps of S_n implies that the expansion of $\varphi_1(s)$ in a Laurent series in a vicinity $|s| > r_2$ of infinity has no principal part,

$$\varphi_1(s) = \widehat{q}_{(0,0)}^{(1,0)} + \frac{\widehat{q}_{(0,-1)}^{(1,0)}}{s} + \frac{\widehat{q}_{(0,-2)}^{(1,0)}}{s^2} + \cdots.$$

Therefore, (as one can show by means of the inequalities $\varphi_1(r_2) < R_1$ and $\varphi_1(s) < R_1$ for $|s| > r_2$) the expansion of $\Phi_1^{(\alpha_1,\alpha_2)}(\varphi_1(s))$ in a Laurent series in the domain $|s| > r_2$ has no principal part. Theorem 3 is proved.

7. Integral representation for $q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)}$ in the case $h(s_1, s_2) = h_0(s_2) + s_1 h_1(s_2)$. Put $h(s_1, s_2) = h_0(s_2) + s_1 h_1(s_2)$, where $h_0(s_2)$, $h_1(s_2)$ are positive generating functions. Figure 3, case (a) shows possible jumps of S_n.

LEMMA 5. *Let $h(s_1, s_2) = h_0(s_2) + s_1 h_1(s_2)$ and, for the probability distribution $\{p_{\gamma_1\gamma_2}, (\gamma_1, \gamma_2) \in \mathbf{N}^2\}$, let there be probabilities $p_{0\gamma_2} > 0$, $p_{\gamma_1\gamma_2} > 0$ with $\gamma_2 \geqq 2$ and $p_{\gamma_1 0} > 0$. Then $\varphi_1(s)$ is an analytic function in the ring $r_2 < |s| < R_2$, $0 \leqq r_2 < 1 < R_2 \leqq \infty$, and $\lim_{s \downarrow r_2} \varphi_1(s) = +\infty$, $\lim_{s \uparrow R_2} \varphi_1(s) = +\infty$; $\varphi_2(s)$ is an analytic function in the domain $|s| > r_1$, $0 < r_1 \leqq 1$, and the value $\widetilde{r}_2 = \varphi_2(r_1)$ belongs to the interval (r_2, R_2).*

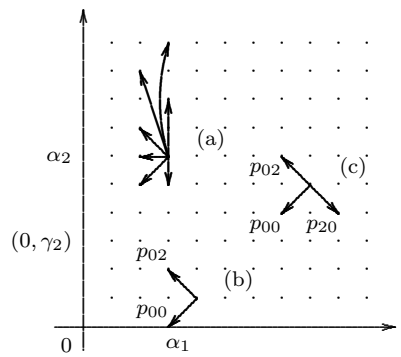

FIG. 3

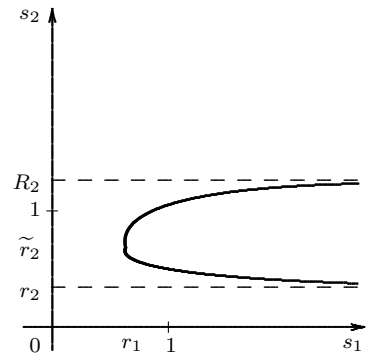

FIG. 4

Under the conditions of Lemma 5 S_L is a two-dimensional lattice. The proof of Lemma 5 is similar to those of Lemmas 2 and 3. One needs to use the expression $\varphi_1(s) = h_0(s)/(s - h_1(s))$ which follows from the characteristic equation. The respective set Q^+ is shown in Figure 4.

Throughout the paper we use the following notation. The characteristic equation $h(s_1, s_2) - s_1 s_2 = 0$ specifies a multivalued function of complex argument. By $\widetilde{\varphi}_2(s)$ we denote the single-valued branch of the function in the domain $|s| > r_1$ considered in Lemma 5 (therefore, $\widetilde{\varphi}_2(s) = \varphi_2^{(0,1)}(s)$ by (11)). The same as in [2, Chap. 5, section 4] one can demonstrate that $s = r_1$ is an isolated branch point of order two. On account of Lemma 5 one can show that there is the second single-valued branch $\overline{\varphi}_2(s)$ in the domain $|s| > r_1$ specified by the condition $\lim_{s \downarrow r_1} \overline{\varphi}_2(s) = \widetilde{r}_2$. Denote by $\varphi_2(s)$ the two-valued branch in the domain $\{|s| > r_1\} \cup \{0 < |s - r_1| < \varepsilon\}$ constituted by the branches $\widetilde{\varphi}_2(s)$, $\overline{\varphi}_2(s)$ and the analytic continuation of the branches to the ring $0 < |s - r_1| < \varepsilon$ ($\varepsilon > 0$ is sufficiently small and, therefore, the circle $|s - r_1| < \varepsilon$ contains no other singular points except r_1).

Set $\overline{\psi}_{21}(s) = \overline{\varphi}_2(\varphi_1(s))$ for $r_2 < |s| < \widetilde{r}_2$ and $\widetilde{\psi}_{21}(s) = \widetilde{\varphi}_2(\varphi_1(s))$ for $\widetilde{r}_2 < |s| < R_2$ (correctness of the definition follows from the inequality $|\varphi_1(s)| > r_1$ for $r_2 < |s| < R_2$). Clearly, $h(\varphi_1(s), \overline{\psi}_{21}(s)) - \varphi_1(s)\,\overline{\psi}_{21}(s) = 0$ and $h(\varphi_1(s), \widetilde{\psi}_{21}(s)) - \varphi_1(s)\,\widetilde{\psi}_{21}(s) = 0$. The equation $h(\varphi_1(s), s_2) - \varphi_1(s)\,s_2 = 0$ with $h(s_1, s_2) = h_0(s_2) + s_1 h_1(s_2)$ and $\varphi_1(s) = h_0(s)/(s - h_1(s))$ takes the simple form

$$\frac{h_0(s)}{s - h_1(s)} = \frac{h_0(s_2)}{s_2 - h_1(s_2)}.$$

Applying to the last equation the implicit function theorem to find singular points we establish that the analytic continuation of $\overline{\psi}_{21}(s)$ to the domain $r_2 < |s| < R_2$ coincides with $\widetilde{\psi}_{21}(s)$ for $\widetilde{r}_2 < |s| < R_2$. Moreover, there exists an analytic continuation of the function to the domain $|s| > r_2$, which we denote by $\psi_{21}(s)$ (the relationship $h(\varphi_1(s), \psi_{21}(s)) - \varphi_1(s)\,\psi_{21}(s) = 0$ is valid for $\psi_{21}(s)$). Clearly, $\widetilde{\psi}_{21}(\overline{\psi}_{21}(s)) = s$ for $r_2 < |s| < \widetilde{r}_2$ and $\overline{\psi}_{21}(\widetilde{\psi}_{21}(s)) = s$ for $\widetilde{r}_2 < |s| < R_2$. Consequently, $\psi_{21}(\psi_{21}(s)) = s$ for $|s| > r_2$ by the conservation principle for functional equations; i.e., $\psi_{21}(s)$ is a univalent function for $|s| > r_2$ (according to the univalency criterion of functions given in [10, Chap. 5, section 1]). Hence it follows that the representation of $\psi_{21}(s)$ as a Laurent series in the domain $|s| > r_2$ has no principal part ([10, Chap. 5, section 1, Corollary 1]; compare with section 3 of the proof of Theorem 5.1 [10, Chap. 5]).

In what follows, the derivative with respect to s_1 of a function $H(s_1,s_2)$ in two variables is denoted as $(H(s_1,s_2))'_{s_1} = \partial H(s_1,s_2)/\partial s_1$.

THEOREM 4. *Let $h(s_1,s_2) = h_0(s_2) + s_1 h_1(s_2)$ and the conditions of Lemma 5 hold. Then the absorption probability equals* ($\alpha_1 \in \mathbf{N}$, $\alpha_2 = 1,2,\dots$)

$$q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)} = \frac{1}{2\pi i}\int_{\theta_\rho^+}\left(\frac{1}{2\pi i}\int_\Theta s_1^{\alpha_1}\varphi_2^{\alpha_2}(s_1)\,\frac{(h(s_1,s_2)-s_1s_2)'_{s_1}}{h(s_1,s_2)-s_1s_2}\,ds_1\right)\frac{ds_2}{s_2^{\gamma_2+1}}, \tag{22}$$

$\gamma_2 = 1,2,\dots$, *where $r_2 < \rho < \widetilde{r}_2$, and the integration contour Θ is described later.*

Proof. The branch $\overline{\varphi}_2(s)$ is obtained as an analytic continuation of the branch $\widetilde{\varphi}_2(s)$ when the branch point r_1 is bypassed along the curve $\theta(t) = r_1 + \varepsilon_1 e^{it}$, $0 \leqq t \leqq 2\pi$, where $0 < \varepsilon_1 < \varepsilon$. If ε_1 is sufficiently small, then the values of $\varphi_2(s)$ on $\theta(t)$ belong to the domain $|s| < R_2$ of analyticity of the function $\Phi_2^{(\alpha_1,\alpha_2)}(s)$. The condition $h(s_1,s_2) = h_0(s_2) + s_1h_1(s_2)$ on the jumps of the random walk S_n implies that $\Phi_1^{(\alpha_1,\alpha_2)}(s)$ is a polynomial, i.e., $\Phi_1^{(\alpha_1,\alpha_2)}(s)$ is analytic in the whole complex plane. Therefore, by the conservation criterion for functional equations, $\overline{\varphi}_2(s)$ satisfies functional equation (16):

$$\Phi_1^{(\alpha_1,\alpha_2)}(s) = s^{\alpha_1}\overline{\varphi}_2^{\alpha_2}(s) - \Phi_2^{(\alpha_1,\alpha_2)}\big(\overline{\varphi}_2(s)\big) + \Phi_2^{(\alpha_1,\alpha_2)}(0), \qquad |s| > r_1.$$

Substituting the last expression in (17) we get ($\Phi_1^{(\alpha_1,\alpha_2)}(0) = \Phi_2^{(\alpha_1,\alpha_2)}(0)$)

$$\Phi_2^{(\alpha_1,\alpha_2)}(s) = \varphi_1^{\alpha_1}(s)\,s^{\alpha_2} - \varphi_1^{\alpha_1}(s)\,\overline{\varphi}_2^{\alpha_2}\big(\varphi_1(s)\big) + \Phi_2^{(\alpha_1,\alpha_2)}\Big(\overline{\varphi}_2\big(\varphi_1(s)\big)\Big) \tag{23}$$

for $r_2 < |s| < \widetilde{r}_2$. Substituting (23) into the integral representation (20) for the desired probabilities gives

$$\begin{aligned} q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)} &= \frac{1}{2\pi i}\int_{\theta_\rho^+}\Big(\varphi_1^{\alpha_1}(s)\,s^{\alpha_2} - \varphi_1^{\alpha_1}(s)\,\overline{\varphi}_2^{\alpha_2}\big(\varphi_1(s)\big)\Big)\frac{ds}{s^{\gamma_2+1}} \\ &\quad + \frac{1}{2\pi i}\int_{\theta_\rho^+}\Phi_2^{(\alpha_1,\alpha_2)}\Big(\overline{\varphi}_2\big(\varphi_1(s)\big)\Big)\frac{ds}{s^{\gamma_2+1}}, \qquad \gamma_2 = 1,2,\dots, \end{aligned} \tag{24}$$

where $r_2 < \rho < \widetilde{r}_2$. The second integral equals zero, since we have mentioned that the function $\overline{\varphi}_2(\varphi_1(s))$ admits an analytical continuation to the domain $|s| > r_2$ and its Laurent expansion in a vicinity of infinity has no principal part. Therefore, the Laurent series of $\Phi_2^{(\alpha_1,\alpha_2)}(\overline{\varphi}_2(\varphi_1(s)))$ in the domain $|s| > r_2$ has no principal part.

Using the Cauchy integral formula we obtain the equality (here $\tilde{\theta}_\rho$ is the circumference $|s_1 - \varphi_1(s)| = \rho$ with radius $\rho > 0$ being small enough for $\tilde{\theta}_\rho$ to belong to the domain $|s_1| > r_1$)

$$\begin{aligned} \varphi_1^{\alpha_1}(s)\,s^{\alpha_2} - \varphi_1^{\alpha_1}(s)\overline{\varphi}_2^{\alpha_2}(\varphi_1(s)) &= \frac{1}{2\pi i}\int_{\tilde{\theta}_\rho^+} s_1^{\alpha_1}\widetilde{\varphi}_2^{\alpha_2}(s_1)\,\frac{ds_1}{s_1-\varphi_1(s)} \\ &\quad - \frac{1}{2\pi i}\int_{\tilde{\theta}_\rho^+} s_1^{\alpha_1}\overline{\varphi}_2^{\alpha_2}(s_1)\,\frac{ds_1}{s_1-\varphi_1(s)} = \frac{1}{2\pi i}\int_\Theta s_1^{\alpha_1}\varphi_2^{\alpha_2}(s_1)\,\frac{ds_1}{s_1-\varphi_1(s)} \end{aligned} \tag{25}$$

(taking into account Lemma 5 it is not difficult to show that $\widetilde{\varphi}_2(\varphi_1(s)) \equiv s$ for $r_2 < |s| < \widetilde{r}_2$), where Θ denotes the following Pochhammer contour. The closed contour starts at a point $r_0 = r_1 + \rho_1$, $0 < \rho_1 < \varepsilon$, of the real line and consists of a simple curve θ^+ lying in the domain $|s_1| > r_1$ and connecting r_0 with a point of $\tilde{\theta}_\rho$; the circumference $\tilde{\theta}_\rho^+$; the curve θ^-; the circumference $\hat{\theta}_{\rho_1}^+$: $|s_1 - r_1| = \rho_1$ of radius ρ_1; the curve θ^+; the circumference $\tilde{\theta}_\rho^-$; the curve θ^-; the circumference $\hat{\theta}_{\rho_1}^-$. Going around Θ

the function $\varphi_2(s_1)$ takes at point r_0 its initial value. Substituting (25) in (24) gives (the variable s in (25) is denoted by s_2)

$$q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)} = \frac{1}{2\pi i}\int_{\theta_\rho^+}\left(\frac{1}{2\pi i}\int_{\Theta} s_1^{\alpha_1}\varphi_2^{\alpha_2}(s_1)\,\frac{ds_1}{s_1-\varphi_1(s_2)}\right)\frac{ds_2}{s_2^{\gamma_2+1}},$$

where $r_2<\rho<\widetilde{r}_2$. Letting

$$\varphi_1(s_2)=\frac{h_0(s_2)}{s_2-h_1(s_2)}$$

in the fraction of the integrand

$$\frac{1}{s_1-\varphi_1(s_2)}=\frac{s_2-h_1(s_2)}{s_1s_2-s_1h_1(s_1)-h_0(s_2)}=\frac{(h(s_1,s_2)-s_1s_2)'_{s_1}}{h(s_1,s_2)-s_1s_2}$$

we get (22). Theorem 4 is proved.

Example 1. Let $h(s_1,s_2)=p_{02}s_2^2+p_{00}$ (Figure 3, case (b)). Then the probability $q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)}$ may be viewed as the hitting probability (for a random walk on the set $\mathbf{N}=\{0,1,\dots\}$ with probability p_{00} of a jump to the left, probability p_{02} of a jump to the right, and absorption at the boundary 0) from the initial state α_2 in the state γ_2 at the α_1th step of the walk. This random walk was considered in [7, Chap. 14, Problem 20]. Formulas (4.8) (see [7, Chap. 14]) show that it is necessary to consider the branches of the multivalued function specified by the equation $h(s_1,s_2)-s_1s_2=0$. Transforming the results of [7] for $q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)}$ to the form (22) gives the following representation:

$$q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)} = \frac{1}{2\pi i}\int_{\theta_\rho^-} s^{\alpha_1}\widetilde{\varphi}_2^{\alpha_2}(s)\left(-\frac{2}{\gamma_2}\right)dT_{\gamma_2}(s), \tag{26}$$

where $\rho>r_1$, $\widetilde{\varphi}_2(s)=(s-\sqrt{s^2-4p_{00}p_{02}})/(2p_{02})$ $(r_1=2\sqrt{p_{00}p_{02}})$,

$$T_1(s)=\frac{s}{2p_{00}},\quad T_2(s)=\frac{s^2-2p_{00}p_{02}}{2p_{00}^2},\quad T_3(s)=\frac{s^3-3p_{00}p_{02}s}{2p_{00}^3},\ \dots$$

are the Chebyshev polynomials specified on the interval $[-2\sqrt{p_{00}p_{02}},2\sqrt{p_{00}p_{02}}]$ by the expansion $(-2\sqrt{p_{00}p_{02}}<s_1<2\sqrt{p_{00}p_{02}},\ |s_2|<\sqrt{p_{00}/p_{02}})$

$$\log(h(s_1,s_2)-s_1s_2)=\log(p_{02}s_2^2+p_{00}-s_1s_2)=\log p_{00}+\sum_{\gamma_2=1}^{\infty}\left(-\frac{2}{\gamma_2}\right)T_{\gamma_2}(s_1)\,s_2^{\gamma_2}$$

(see [16, Chap. 10, section 11, relation (30)]). Let us calculate the probability $q_{(0,3)}^{(4,1)}$ by means of (26) using the expansion of $\widetilde{\varphi}_2(s)$ in the Laurent series in the vicinity $|s|>r_1$ of $s=\infty$ (see [9, Chap. IV, section 24, Example 11]):

$$\begin{aligned}
q_{(0,3)}^{(4,1)} &= \frac{1}{2\pi i}\int_{\theta_\rho^-} s^4\,\widetilde{\varphi}_2(s)\left(-\frac{2}{3}\right)dT_3(s)\\
&= \frac{1}{2\pi i}\int_{\theta_\rho^-} s^4\cdot s\,\frac{1-\sqrt{1-4p_{00}p_{02}/s^2}}{2p_{02}}\left(-\frac{2}{3}\right)d\left(\frac{s^3-3p_{00}p_{02}s}{2p_{00}^3}\right)\\
&= \frac{1}{2\pi i}\int_{\theta_\rho^+} s^4\left(\frac{p_{00}}{s}+\frac{p_{00}^2p_{02}}{s^3}+\frac{2p_{00}^3p_{02}^2}{s^5}+\frac{5p_{00}^4p_{02}^3}{s^7}+\cdots\right)\left(\frac{s^2-p_{00}p_{02}}{p_{00}^3}\right)ds\\
&= \frac{5p_{00}^4p_{02}^3-2p_{00}^4p_{02}^3}{p_{00}^3}=3p_{00}p_{02}^3.
\end{aligned}$$

The same value of $q_{(0,3)}^{(4,1)}$ can be obtained by analyzing Figure 3.

8. Integral representation for the probabilities $q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)}$ for the case $h(s_1,s_2)=p_{20}s_1^2+p_{02}s_2^2+p_{00}$. Let ω and ω' be positive numbers. Put $h=\exp\{-\pi\omega'/\omega\}$, $z=\exp\{\pi iu/(2\omega)\}$. Let us define a function which is periodic in two ways by the equality ($\gamma=1,2,\dots$)

$$f_\gamma(u)=\left(\frac{\pi i}{\omega}\right)^\gamma\left\{\frac{1}{(z-z^{-1})^\gamma}+\sum_{r=1}^\infty\left(\frac{h^rz^{-1}}{1-h^{2r}z^{-2}}\right)^\gamma+\sum_{r=1}^\infty\left(\frac{-h^rz}{1-h^{2r}z^2}\right)^\gamma\right\}. \tag{27}$$

The periods of the functions for odd γ are 4ω and $2i\omega'$, and for even γ, 2ω and $2i\omega'$. Convergence of the series having form (27) was investigated in [13, Chap. 2, section 12]. It is easy to derive from (27) the expansion in a Fourier series

$$\begin{aligned}f_\gamma(u)=\left(\frac{\pi}{\omega}\right)^\gamma\Bigg\{&\frac{1}{2^\gamma}\operatorname{cosec}^\gamma\left(\frac{\pi u}{2\omega}\right)+(-1)^{[\gamma/2]}2\\&\times\sum_{n=1}^\infty C_{n+\gamma-1}^{\gamma-1}\frac{h^{2n+\gamma}}{1-h^{2n+\gamma}}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\left(\frac{\pi u}{2\omega}(2n+\gamma)\right)\Bigg\},\end{aligned} \tag{28}$$

where cos is applied for even γ and sin for odd γ. Expansion (28) is valid in the stripe $\operatorname{Im}u<2\omega'$, except the poles of the first summand (see [12, section 22.61, Problem 20.35]). Comparing (28) with the known Fourier expansions of Jacobi elliptic functions (see [12, section 22.61, Problem 22.57]) we get

$$f_\gamma(u)=\frac{c_\gamma}{(\operatorname{sn}cu)^\gamma}+\frac{c_{\gamma-2}}{(\operatorname{sn}cu)^{\gamma-2}}+\dots+c_0,$$

where c, $c_\gamma,\dots$ are some constants. In particular, $f_2(u)$ coincides up to a constant summand with the Weierstrass elliptic function [13].

THEOREM 5. *Let $h(s_1,s_2)=p_{20}s_1^2+p_{02}s_2^2+p_{00}$. Put*

$$C=\frac{1-\sqrt{1-4p_{20}p_{02}}}{1+\sqrt{1-4p_{20}p_{02}}},\qquad C_0=\frac{p_{20}p_{00}}{1-4p_{20}p_{02}};$$

$f_{\gamma_2}(u)$ is the elliptic function (27) *with $\omega=\pi\sqrt{C_0}$, $\omega'=-\sqrt{C_0}\log C$, and $T=\{u=x+iy,\ 0\leqq x\leqq 4\omega,\ y=-\omega'\}$ is the interval with orientation corresponding to the growth of x. The absorption probability is* ($(\alpha_1,\alpha_2)\in\mathbf{N}^2$)

$$q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)}=\frac{1}{2\pi i}\int_T s_1^{\alpha_1}\left(e^{\pi iu/(2\omega)}\right)s_2^{\alpha_2}\left(e^{\pi iu/(2\omega)}\right)\frac{(-1)^{\gamma_2}}{-\gamma_2}\,df_{\gamma_2}(u), \tag{29}$$

$\gamma_2=1,2,\dots$, *where the functions*

$$s_1(z)=\frac{1-\sqrt{1-4p_{20}p_{02}}}{2p_{20}}\,i\sqrt{C_0}\left(z-\frac{1}{Cz}\right),\quad s_2(z)=i\sqrt{C_0}\left(z-\frac{1}{z}\right) \tag{30}$$

are a uniformization of the Riemann surface (6).

Figure 3, case (c) shows possible jumps of S_n. An example of calculating the absorption probability by formula (29) is given in [20].

Proof. Equation (4) for the exponential generating function of the probabilities $q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)}$ takes the form

$$\left(p_{20}\frac{\partial^2}{\partial z_1^2}+p_{02}\frac{\partial^2}{\partial z_2^2}+p_{00}-\frac{\partial^2}{\partial z_1\partial z_2}\right)g_{(0,\gamma_2)}(z_1,z_2)=0 \tag{31}$$

with the boundary conditions (5); the corresponding characteristic equation has the form

$$p_{20}s_1^2 + p_{02}s_2^2 + p_{00} - s_1 s_2 = 0. \tag{32}$$

A) *Construction of a nonclosed solution of the Darboux–Picard problem* (31), (5). Equation (31) of the second order belongs to the hyperbolic type (see [14], [15]). We find the general integrals of the equations for characteristics. Introducing new independent variables x, y instead of z_1, z_2,

$$\begin{aligned} x &= \frac{1-\sqrt{1-4p_{20}p_{02}}}{2p_{20}}\, z_1 + z_2, \\ y &= \frac{p_{00}p_{20}}{1-4p_{20}p_{02}}\left(\frac{1+\sqrt{1-4p_{20}p_{02}}}{2p_{20}}\, z_1 + z_2\right), \end{aligned} \tag{33}$$

we reduce (31) to the canonical form

$$\frac{\partial^2 g_{(0,\gamma_2)}}{\partial x \partial y} = g_{(0,\gamma_2)} \tag{34}$$

(the so-called "telegraph equation"). Under transformation, (33) maps the lines $z_1 = 0$ and $z_2 = 0$ into the lines $y = C_0 x$ and $y = (C_0/C)\, x$, respectively; the boundary conditions (5) transform to the respective conditions for (34):

$$g_{(0,\gamma_2)}(x,y)\,\big|_{y=C_0 x} = \frac{x^{\gamma_2}}{\gamma_2!}, \quad g_{(0,\gamma_2)}(x,y)\,\big|_{y=C_0 x/C} = 0. \tag{35}$$

Thus, we solve the Darboux–Picard problem for (34): conditions (35) are specified on two monotone curves starting from point $(0,0)$ and located within the characteristic angle $x = 0$, $y = 0$ with vertex at the same point $(0,0)$. We know from the theory of partial differential equations of the second order of hyperbolic type that a solution of problem (34), (35) exists and is unique (compare the analyticity condition (3) for the desired solution). To construct the needed solution of problem (34), (35) we make use of the general solution for the telegraph equation [14, Problem 30]:

$$\begin{aligned} g_{(0,\gamma_2)}(x,y) = {} & \int_0^x \psi_1(t)\, J_0\Big(2i\,\sqrt{y(x-t)}\,\Big)\, dt + \int_0^y \psi_2(t)\, J_0\Big(2i\,\sqrt{x(y-t)}\,\Big)\, dt \\ & + g_{(0,\gamma_2)}(0,0)\, J_0(2i\,\sqrt{xy}), \end{aligned} \tag{36}$$

where $\psi_1(t)$ and $\psi_2(t)$ are arbitrary functions and $J_0(t)$ is the Bessel function of order zero. Expression (36) is the Riemann formula [15] for the solution of (34).

For $\gamma_2 = 1, 2, \ldots$, we have $g_{(0,\gamma_2)}(0,0) = 0$. We find functions $\psi_1(t)$ and $\psi_2(t)$ meeting conditions (35). Substituting the second of conditions (35) into (36) gives

$$\begin{aligned} g_{(0,\gamma_2)}(x,y)\,\Big|_{y=C_0 x/C} &= \int_0^x \psi_1(t)\, J_0\left(2i\,\sqrt{\frac{C_0}{C}\, x(x-t)}\right) dt \\ &\quad + \int_0^{C_0 x/C} \psi_2(t)\, J_0\left(2i\,\sqrt{x\left(\frac{C_0}{C}\, x - t\right)}\right) dt \\ &= \int_0^x \left(\psi_1(t) + \frac{C_0}{C}\, \psi_2\left(\frac{C_0}{C}\, t\right)\right) J_0\left(2i\,\sqrt{\frac{C_0}{C}\, x(x-t)}\right) dt = 0. \end{aligned}$$

Hence $\psi_2(t) = -(C/C_0)\,\psi_1((C/C_0)\,t)$. Replacing $\psi_2(t)$ in (36) by the last expression and then substituting the first of conditions (35) into (36) we obtain

$$g_{(0,\gamma_2)}(x,y)\Big|_{y=C_0x} = \int_0^x \psi_1(t)\,J_0\Big(2i\,\sqrt{C_0x(x-t)}\,\Big)\,dt - \int_0^{C_0x} \frac{C}{C_0}\,\psi_1\Big(\frac{C}{C_0}\,t\Big)\,J_0\Big(2i\,\sqrt{x(C_0x-t)}\Big)\,dt = \frac{x^{\gamma_2}}{\gamma_2!}.$$

Or, changing the variable in the second summand,

$$\frac{x^{\gamma_2}}{\gamma_2!} = \int_0^x \big(\psi_1(t) - C\psi_1(Ct)\big)\,J_0\Big(2i\,\sqrt{C_0x(x-t)}\,\Big)\,dt. \tag{37}$$

We look for $\psi_1(t)$ in the form of a series $\psi_1(t) = \sum_{k=0}^{\infty} a_{\gamma_2-1+2k}t^{\gamma_2-1+2k}$. Then equality (37) takes the form

$$\frac{x^{\gamma_2}}{\gamma_2!} = \int_0^x \left(\sum_{k=0}^{\infty} a_{\gamma_2-1+2k}(1-C^{\gamma_2+2k})\,t^{\gamma_2-1+2k}\right) J_0\Big(2i\,\sqrt{C_0x(x-t)}\,\Big)\,dt.$$

Taking into account the values of the integrals (compare with [17, Integral 6.567.1])

$$\int_0^x t^{\gamma_2-1+2k}J_0\Big(2i\,\sqrt{C_0x(x-t)}\,\Big)\,dt = \frac{(\gamma_2-1+2k)!}{(i\sqrt{C_0})^{\gamma_2+2k}}\,J_{\gamma_2+2k}\left(2i\,\sqrt{C_0}x\right), \tag{38}$$

where $k = 0, 1, \dots$, we see that

$$\frac{x^{\gamma_2}}{\gamma_2!} = \sum_{k=0}^{\infty} a_{\gamma_2-1+2k}\frac{(1-C^{\gamma_2+2k})(\gamma_2-1+2k)!}{(i\sqrt{C_0})^{\gamma_2+2k}}\,J_{\gamma_2+2k}\left(2i\,\sqrt{C_0}x\right).$$

Comparing the last expression with the Neumann series [16, section 7.10.1]

$$x^{\gamma_2} = \sum_{k=0}^{\infty} \frac{(\gamma_2+2k)(\gamma_2+k-1)!}{k!}\,J_{\gamma_2+2k}(2x),$$

we find the desired expressions for a_{γ_2-1+2k}:

$$a_{\gamma_2-1+2k} = \frac{(-1)^kC_0^k}{1-C^{\gamma_2+2k}}\,\frac{(\gamma_2+2k)(\gamma_2+k-1)!}{\gamma_2!\,k!(\gamma_2-1+2k)!}, \qquad k = 0, 1, \dots .$$

Substituting the functions $\psi_1(t)$ and $\psi_2(t)$ found in formula (36), we obtain, by calculating the respective integrals of form (38), an expression for $g_{(0,\gamma_2)}(x,y)$ as a series with respect to the Bessel functions of integer orders:

$$g_{(0,\gamma_2)}(x,y) = \sum_{k=0}^{\infty} \frac{C_0^k}{1-C^{\gamma_2+2k}}\,\frac{(\gamma_2+2k)(\gamma_2+k-1)!}{i^{\gamma_2}\gamma_2!\,k!} \times \left(\left(\sqrt{\frac{x}{y}}\right)^{\gamma_2+2k} - \left(\frac{C}{C_0}\,\sqrt{\frac{y}{x}}\right)^{\gamma_2+2k}\right) J_{\gamma_2+2k}\left(2i\,\sqrt{xy}\,\right). \tag{39}$$

Let us replace the Bessel functions with the integrals

$$J_n(2t) = \frac{1}{2\pi i}\oint_{0+} e^{t(u-1/u)}\,\frac{du}{u^{n+1}}, \qquad n = 0, \pm1, \pm2, \dots .$$

Then, making the substitution $u = \sqrt{C_0 x/y}\, z$ we get

$$\left(\sqrt{\frac{x}{y}}\right)^{\gamma_2+2k} J_{\gamma_2+2k}\left(2i\sqrt{xy}\right)$$
$$= \frac{1}{(\sqrt{C_0})^{\gamma_2+2k}} \frac{1}{2\pi i} \oint_{0+} \exp\left\{i\sqrt{C_0}\left(xz - \frac{y}{C_0 z}\right)\right\} \frac{dz}{z^{\gamma_2+2k+1}}.$$

Putting $u = C\sqrt{y/(C_0 x)}z$ gives

$$\left(\frac{C}{C_0}\sqrt{\frac{y}{x}}\right)^{\gamma_2+2k} J_{\gamma_2+2k}\left(2i\sqrt{xy}\right)$$
$$= \frac{1}{(\sqrt{C_0})^{\gamma_2+2k}} \frac{1}{2\pi i} \oint_{0+} \exp\left\{i\sqrt{C_0}\left(\frac{Cyz}{C_0} - \frac{x}{Cz}\right)\right\} \frac{dz}{z^{\gamma_2+2k+1}}.$$

Substituting the last expressions in (39) and replacing x and y with formulas (33) we arrive at a nonclosed solution of problem (31), (5):

$$g_{(0,\gamma_2)}(z_1, z_2) = \sum_{k=0}^{\infty} \frac{1}{(i\sqrt{C_0})^{\gamma_2}(1 - C^{\gamma_2+2k})} \frac{(\gamma_2+2k)(\gamma_2+k-1)!}{\gamma_2!\, k!} \frac{1}{2\pi i}$$
$$\times \oint_{0+} \left[\exp\left\{z_1 \frac{1-\sqrt{1-4p_{20}p_{02}}}{2p_{20}}\, i\sqrt{C_0}\left(z - \frac{1}{Cz}\right) + z_2 i\sqrt{C_0}\left(z - \frac{1}{z}\right)\right\}\right.$$
$$- \exp\left\{z_1 \frac{1-\sqrt{1-4p_{20}p_{02}}}{2p_{20}}\, i\sqrt{C_0}\left(z - \frac{1}{Cz}\right)\right.$$
$$\left.\left. + z_2 i\sqrt{C_0}\left(Cz - \frac{1}{Cz}\right)\right\}\right] \frac{dz}{z^{\gamma_2+2k+1}}. \tag{40}$$

B) *Construction of a closed solution of problem* (31), (5). One can justify for series (40) the correctness of permuting the order of summation and integration or summation and differentiation. The following transformations are well defined given that $C < 1$ and $|z| > 1$ ($\gamma_2 = 1, 2, \dots$):

$$\sum_{k=0}^{\infty} \frac{1}{1 - C^{\gamma_2+2k}} \frac{(\gamma_2+k-1)!}{(\gamma_2-1)!\, k!} \frac{1}{z^{\gamma_2+2k}} = \sum_{k=0}^{\infty} \frac{(\gamma_2+k-1)!}{(\gamma_2-1)!\, k!} \frac{1}{z^{\gamma_2+2k}} \sum_{r=0}^{\infty} C^{r(\gamma_2+2k)}$$
$$= \sum_{r=0}^{\infty} \left(\frac{C^r}{z}\right)^{\gamma_2} \sum_{k=0}^{\infty} \frac{(\gamma_2+k-1)!}{(\gamma_2-1)!\, k!} \left(\frac{C^{2r}}{z^2}\right)^k = \sum_{r=0}^{\infty} \left(\frac{C^r}{z}\right)^{\gamma_2} \left(1 - \frac{C^{2r}}{z^2}\right)^{-\gamma_2}.$$

Then expression (40) for $g_{(0,\gamma_2)}(z_1, z_2)$ takes the form

$$g_{(0,\gamma_2)}(z_1, z_2) = \frac{1}{2\pi i}$$
$$\times \oint_{\theta+} \left[\exp\left\{z_1 \frac{1-\sqrt{1-4p_{20}p_{02}}}{2p_{20}}\, i\sqrt{C_0}\left(z - \frac{1}{Cz}\right) + z_2 i\sqrt{C_0}\left(z - \frac{1}{z}\right)\right\}\right.$$
$$\left. - \exp\left\{z_1 \frac{1-\sqrt{1-4p_{20}p_{02}}}{2p_{20}}\, i\sqrt{C_0}\left(z - \frac{1}{Cz}\right) + z_2 i\sqrt{C_0}\left(Cz - \frac{1}{Cz}\right)\right\}\right]$$
$$\times \frac{1}{-\gamma_2}\, d\left(\frac{1}{(i\sqrt{C_0})^{\gamma_2}} \sum_{r=0}^{\infty} \left(\frac{C^r z^{-1}}{1 - C^{2r} z^{-2}}\right)^{\gamma_2}\right),$$

where the circumference $|z| = 1/\sqrt{C}$ is taken as the integration contour θ. Writing $-1/(Cz)$ for z in the second summand we get

$$g_{(0,\gamma_2)}(z_1,z_2) = \frac{1}{2\pi i}\oint_{\theta+} e^{z_1 s_1(z)+z_2 s_2(z)} \times \frac{(-1)^{\gamma_2}}{-\gamma_2}\, d\left(\left(\frac{i}{\sqrt{C_0}}\right)^{\gamma_2}\left\{\sum_{r=0}^{\infty}\left(\frac{C^r z^{-1}}{1-C^{2r}z^{-2}}\right)^{\gamma_2} + \sum_{r=1}^{\infty}\left(\frac{-C^r z}{1-C^{2r}z^2}\right)^{\gamma_2}\right\}\right).$$

Comparing the last expression with (27) we obtain the following closed representation for $g_{(0,\gamma_2)}(z_1,z_2)$:

$$g_{(0,\gamma_2)}(z_1,z_2) = \frac{1}{2\pi i}\int_T e^{z_1 s_1(z)+z_2 s_2(z)}\,\frac{(-1)^{\gamma_2}}{-\gamma_2}\, df_{\gamma_2}(u); \tag{41}$$

by comparing the parameters involved we find the values of the periods: 4ω and $2i\omega'$ of $f_{\gamma_2}(u)$ and the integration interval T given in the statement of the theorem.

Some of the transformations above need a detailed justification of their correctness; however, by direct substitution of (41) into (31) and by checking the validity of conditions (5) we conclude that (41) solves problem (31), (5). The definitions of exponential generating function (2) and expansion (8) yield formula (29) for the probabilities $q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)}$. Theorem 5 is proved.

Remark 2. Comparing (29) with the formulas for $q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)}$ derived in Theorems 2–4 one may guess that $f_{\gamma_2}(u) = (f_1(u) - c_1)^{\gamma_2} + c_{\gamma_2}$, where $c_1, c_2, \dots$ are some constants.

Remark 3. The exponent of the first exponent in solution (40) has the form $z_1 s_1(z) + z_2 s_2(z)$, where $s_1(z)$ and $s_2(z)$ are the functions specified by formulas (30); $s_1(z)$ and $s_2(z)$ are rational functions satisfying the characteristic equation (32) and, therefore, are a uniformization of the Riemann surface (6). It is easy to see that the functions

$$v_n(z_1,z_2) = \frac{1}{2\pi i}\oint_{0+} e^{z_1 s_1(z)+z_2 s_2(z)}\,\frac{dz}{z^{n+1}}, \qquad n = 0, \pm 1, \pm 2, \dots,$$

solve (31). Making a substitution of variable in the integrand of the second summand in (40), one can rewrite series (40) as

$$g_{(0,\gamma_2)}(z_1,z_2) = \sum_{n=-\infty}^{\infty} a_n v_n(z_1,z_2). \tag{42}$$

For $h(s_1,s_2) = p_{20}s_1^2 + p_{02}s_2^2 + p_{10}s_1 + p_{01}s_2 + p_{00}$ or $h(s_1,s_2) = p_{22}s_1^2 s_2^2 + p_{21}s_1^2 s_2 + p_{12}s_1 s_2^2 + p_{20}s_1^2 + p_{02}s_2^2$ one can deduce a nonclosed representation (42) for the solution of the corresponding equation (4) under simple uniformization of the two-sheeted Riemann surface (6). To do this it is necessary to obtain in these cases an integral representation of type (9) for the probabilities $q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)}$ by the methods of section 8.

Remark 4. In the particular case of branching processes we consider here, particles of type T_2 are called final [2]. Denote by $\eta^{(\alpha_1,\alpha_2)}$ the number of final particles when the process dies, i.e., when there are no particles of T_1. Clearly, $\eta^{(\alpha_1,\alpha_2)}$ is a random variable with distribution $\{q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)},\ \gamma_2 = 0,1,\dots;\ \sum_{\gamma_2=0}^{\infty} q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)} \leqq 1\}$. The obtained formulas for $q_{(0,\gamma_2)}^{(\alpha_1,\alpha_2)}$ can be used, in principle, to establish limit theorems for the number of final particles under the conditions $\alpha_1 \to \infty, \alpha_2 \to \infty$ ([2, Chap. 5, section 5], [1], [6]). Such theorems are important for applications.

Acknowledgment. The author thanks A. Kulikovskikh for a number of valuable remarks.

REFERENCES

[1] V. A. Malyshev, *Random Walks. The Wiener–Hopf Equation in a Quadrant of the Plane. Galois Automorphisms*, Moscow State University Publishing House, Moscow, 1970 (in Russian).

[2] B. A. Sewastjanow, *Verzweigungsprozesse*, Akademie-Verlag, Berlin, 1974.

[3] V. A. Malyshev, *Wiener–Hopf equations and their applications in probability theory*, J. Sov. Math., 7 (1977), pp. 129–148.

[4] B. A. Sevastyanov and A. V. Kalinkin, *Random branching processes with interaction of particles*, Sov. Math. Dokl., 25 (1982), pp. 644–646.

[5] Yu. I. Gromak and V. A. Malyshev, *The hitting probability in a finite set for a random walk in a quadrant of the plane with absorption at the boundary*, in Proceedings of International Conference in Probability Theory and Mathematical Statistics: Abstracts of Communications, Vilnius, 1973, pp. 185–186 (in Russian).

[6] A. A. Mogulskii and B. A. Rogozin, *Random walks in the positive quadrant.* I: *Local theorems*; II: *Integral theorems*; III: *Constants in the integral and local theorems*, Siberian Adv. Math., 10 (2000), pp. 34–72, 57–97; 10 (2000), pp. 35–103; 11 (2001), pp. 35–59.

[7] W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. 1, Wiley, New York, 1971.

[8] *Two-sex problem. Encyclopedia of Statistical Sciences*, Vol. 9, Wiley, New York, 1988.

[9] Yu. V. Sidorov, M. V. Fedoryuk, and M. I. Shabunin, *Lectures on the Theory of Functions of a Complex Variable*, Mir, Moscow, 1985.

[10] M. A. Evgrafov, *Analytic Functions*, W. B. Saunders, Philadelphia, 1966.

[11] V. P. Palamodov, *Linear Differential Operators with Constant Coefficients*, Springer-Verlag, New York, Berlin, 1970.

[12] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis. An Introduction to the General Theory of Infinite Processes and of Analytic Functions*: *With an Account of the Principal Transcendental Functions*, 4th ed., Cambridge University Press, Cambridge, UK, 1996.

[13] A. Gurvits, *Theory of Analytic and Elliptic Functions*, GTTI, Leningrad, 1933 (in Russian).

[14] M. M. Smirnov, *Problems on the Equations of Mathematical Physics*, Noordhoff, Gröningen, 1966.

[15] A. V. Bitsadze and D. F. Kalinichenko, *A Collection of Problems on the Equations of Mathematical Physics*, Mir, Moscow, 1983.

[16] G. Beitmen and A. Erdeii, *Higher Transcendental Functions. Vol.* 2. *Bessel Functions*, *Parabolic Cylinder Functions*, *Orthogonal Polynomial*, McGraw–Hill, New York, Toronto, London, 1953.

[17] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals*, *Series*, *and Products*, Academic Press, Boston, MA, 1994.

[18] A. V. Kalinkin, *Extinction probability of a branching process with interaction of particles*, Theory Probab. Appl., 27 (1982), pp. 201–205.

[19] A. V. Kalinkin, *Final probabilities for a random branching process with interaction of particles*, Sov. Math. Dokl., 27 (1983), pp. 493–497.

[20] A. V. Kalinkin, *The exponential generating function method for random walks in a quadrant of the plane*, Dokl. RAS, 275 (2000), pp. 583–587 (in Russian).