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ON THE EXTINCTION PROBABILITY OF A BRANCHING PROCESS WITH TWO KINDS OF INTERACTION OF PARTICLES*

A. V. KALINKIN†

(*Translated by V. A. Vatutin*)

Abstract. A continuous-time Markov branching process with a single type T of particles is considered in which any pair of particles $T+T$ produces offspring independently of all other particles. In addition, any particle of type T also produces children. Representations for the extinction probabilities of this process are obtained under certain assumptions on the offspring distribution. To establish this result we apply the exponential generating function method for the solution of a stationary backward Kolmogorov system of differential equations (see [A. V. Kalinkin, *Theory Probab. Appl.*, 27 (1982), pp. 201–205]).

Key words. branching process with interaction of particles, extinction probabilities, Laplace equation for the exponential generating function, explicit solutions

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1. A branching process with a single type of particles T and two kinds of interaction $\varepsilon^1 = 1$, $\varepsilon^2 = 2$ (see [5]). We consider a time homogeneous Markov process ξ_t, $t \in [0,\infty)$, with the state space $\mathbf{N} = \{0,1,2,\dots\}$, and transition probabilities $P_{ij}(t) = \mathbf{P}\{\xi_t = j \mid \xi_0 = i\}$, $i,j \in \mathbf{N}$. Let the transition probabilities have the following form as $\Delta t \to 0$ ($\lambda \geqq 0$, $\mu \geqq 0$):

$$P_{ij}(\Delta t) = \begin{cases} (p^2_{j-i+2} i(i-1)\,\lambda + p^1_{j-i+1} i\mu)\,\Delta t + o\,(\Delta t) & \text{if } j \geqq i-2, j \neq i, \\ 1 - (i(i-1)\,\lambda + i\mu)\,\Delta t + o\,(\Delta t) & \text{if } j = i, \\ o\,(\Delta t) & \text{if } j \leqq i-2, \end{cases}$$

where $\{p^k_l \geqq 0,\ l \in \mathbf{N};\ \sum_{l=0}^{\infty} p^k_l = 1,\ p^k_k = 0\}$, $k = 1,2$, are given probability distributions. Introduce the generating functions

$$F_i(t;s) = \sum_{j=0}^{\infty} P_{ij}(t) s^j, \quad i \in \mathbf{N}; \qquad h_k(s) = \sum_{l=0}^{\infty} p^k_l s^l, \quad k = 1,2,\ |s| \leqq 1.$$

The second (direct) Kolmogorov system of differential equations for the process ξ_t is equivalent to the following linear partial differential equation (see [5, Thm. 1]):

$$\frac{\partial F_i(t;s)}{\partial t} = \lambda(h_2(s) - s^2)\,\frac{\partial^2 F_i(t;s)}{\partial s^2} + \mu(h_1(s) - s)\,\frac{\partial F_i(t;s)}{\partial s} \tag{1}$$

with the initial condition $F_i(0;s) = s^i$.

A state i is treated as the fact that there are i particles of type T. One may assume that interactions of a type T particle with other type T particles occur after random time intervals. This pair of particles transforms, independently of other particles, into a new group of n particles of type T: $T+T \to nT$, according to the probability distribution $\{p^2_n\}$; if this event occurs, the state of the process changes from i to $i+n-2$. In addition, any of

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†Department of Higher Mathematics, Bauman Moscow State Technical University, 2nd Baumanskaya St., 5, 107005 Moscow, Russia (www.bmstu.ru/facult/fn/kalin/kalinkin.htm).

the existing type T particles transforms after a random time interval into a group consisting of m particles of type T: $T \to mT$, according to the probability distribution $\{p_m^1\}$; if this event occurs, the state of the process changes from i to $i+m-1$.

The process ξ_t belongs to a particular class of Markov processes with a countable number of states introduced by Sevastyanov in [5]. Papers [1] and [2] treat this process as a model of chemical reactions. Letessier and Valent (see [3] and references therein), in connection with some applications in high power physics, constructed explicit nonclosed solutions of equation (1) for various particular cases of $h_1(s)$ and $h_2(s)$.

2. The problem of extinction probabilities. State 0 is absorbing for the Markov process ξ_t. The extinction probabilities, that is, the probabilities of hitting the absorbing state, are equal as follows:

$$q_{i0} = \lim_{t\to\infty} P_{i0}(t), \quad i \in \mathbf{N}. \tag{2}$$

For an ordinary branching process with independent particles in which $\lambda = 0$, $\mu > 0$, we have $q_{i0} = q_1^i$, $i \in \mathbf{N}$, where q_1 is the nearest to zero nonnegative solution of the equation $h_1(s) - s = 0$ (see [4, Chap. 2, section 1]).

For a branching process with pairwise interaction of particles, in which $\lambda > 0$, $\mu = 0$ ($p_0^2 > 0$), we have

$$q_{i0} \sim C_0 q_2^i, \quad i \to \infty, \tag{3}$$

where $C_0 > 0$ and q_2 is the nearest to zero nonnegative solution of the equation $h_2(s) - s^2 = 0$ (see [5] and [7, Corollary 1]).

A branching process ξ_t with $\lambda > 0$ and $\mu > 0$ is characterized by the *criticality parameters* $m_1 = \mu(h_1'(1) - 1)$, $m_2 = \lambda(h_2'(1) - 2)$ (see [4], [5]). The set of all possible variants of the values of these parameters can be divided into the following three principal cases:

1) $m_2 > 0$;
2) $m_2 = 0$, $m_1 > 0$;
3) $m_2 = 0$, $m_1 \leqq 0$, or $m_2 < 0$.

The present paper deals with case 1) and contains expressions for the extinction probabilities (2) under some additional assumptions about the offspring distribution $\{p_m^1,\ m \in \mathbf{N}\}$, $\{p_n^2,\ n \in \mathbf{N}\}$. It is shown that the asymptotic behavior of the extinction probabilities q_{i0} as $i \to \infty$ may have a form other than (3).

In case 3), one may suppose that $q_{i0} = 1$ for all $i \in \mathbf{N}$.

3. Differential equations for the exponential generating function of the extinction probabilities. Introduce the exponential generating functions

$$G_j(t;z) = \sum_{i=0}^{\infty} \frac{z^i}{i!} P_{ij}(t), \quad j \in \mathbf{N}; \qquad g_0(z) = \sum_{i=0}^{\infty} \frac{z^i}{i!} q_{i0}, \tag{4}$$

and the linear differential operators

$$h_k\left(\frac{d}{dz}\right) = \sum_{l=0}^{\infty} p_l^k \frac{d^l}{dz^l}, \quad k = 1, 2.$$

The first (backward) Kolmogorov system of differential equations for the process ξ_t is equivalent to the linear partial differential equation ([5, Thm. 2]; compare with [7, Thm. 1])

$$\frac{\partial G_j(t;z)}{\partial t} = \left[\lambda z^2\left(h_2\left(\frac{\partial}{\partial z}\right) - \frac{\partial^2}{\partial z^2}\right) + \mu z\left(h_1\left(\frac{\partial}{\partial z}\right) - \frac{\partial}{\partial z}\right)\right] G_j(t;z)$$

with the initial condition $G_j(0;z) = z^j/j!$.

One can show, similarly to Theorem 2 in [7], that $g_0(z) = \lim_{t\to\infty} G_0(t;z)$ and that $g_0(z)$ satisfies the stationary backward Kolmogorov equation

$$\left[\lambda z\left(h_2\left(\frac{d}{dz}\right) - \frac{d^2}{dz^2}\right) + \mu\left(h_1\left(\frac{d}{dz}\right) - \frac{d}{dz}\right)\right] g_0(z) = 0 \tag{5}$$

subject to the boundary condition $g_0(0) = 1$. Observe that $g_0(z)$ is an entire function of order not exceeding 1 since

$$|g_0(z)| \leqq \sum_{i=0}^{\infty} \frac{|z|^i}{i!}\,|q_{i0}| \leqq e^{|z|}. \tag{6}$$

4. On the integral representation for the extinction probabilities q_{i0}. Let $h_1(s)$ and $h_2(s)$ be polynomials. Equation (5) is an ordinary differential equation of Laplace's type (see [8, Part I, Chap. 5, section 22.4]) and can be solved by the definite integral method

$$g_0(z) = \int_K e^{zu}\varphi(u)\,du, \tag{7}$$

where

$$\varphi(u) = \frac{1}{h_2(u) - u^2}\exp\left(\frac{\mu}{\lambda}\int_{u_0}^{u}\frac{h_1(v) - v}{h_2(v) - v^2}\,dv\right),$$

and K is a curve in the complex u-plane meeting the condition

$$\int_K \frac{d}{du}\,e^{zu}(h_2(u) - u^2)\varphi(u)\,du = 0.$$

Definition (4) of the exponential generating function $g_0(z)$ and the expansion $e^{zu} = \sum_{i=0}^{\infty} z^i\,u^i/i!$ lead to the following representation for the extinction probabilities

$$q_{i0} = \int_K \frac{u^i}{h_2(u) - u^2}\exp\left(\frac{\mu}{\lambda}\int_{u_0}^{u}\frac{h_1(v) - v}{h_2(v) - v^2}\,dv\right)du, \quad i \in \mathbf{N}.$$

To find the curve K it is necessary to solve a complicated problem related to the analysis of the Riemann surface corresponding to the function $\varphi(u)$ [8]. In the present paper we find q_{i0} for particular cases of $h_1(s)$ and $h_2(s)$.

Observe that function e^z solves equation (5) for any $h_1(s)$ and $h_2(s)$.

5. Integral representation for the extinction probabilities q_{i0} in the case $h_1(s) = p_2^1 s^2 + p_0^1$, $h_2(s) = p_3^2 s^3 + p_1^2 s + p_0^2$, $m_2 > 0$.

LEMMA. *Let*

$$h_1(s) = p_2^1 s^2 + p_0^1, \qquad h_2(s) = p_3^2 s^3 + p_1^2 s + p_0^2,$$

with $p_0^1 + p_0^2 > 0$, $m_2 > 0$, *and* $p_0^2 + p_1^2 > 0$. *Then* $\lim_{i\to\infty} q_{i0} = 0$.

Proof. We restrict ourselves to considering the case $h_2(s) = p_3^2 s^3 + p_1^2 s$. In this situation, ξ_t may have jumps -1 and $+1$ only. Considering this process only at the moments of changing its state, we obtain an "embedded Markov chain" $X_0, X_1, \dots, X_n, \dots$ such that $(i \neq 0)$

$$\mathbf{P}\{X_{n+1} = i - 1 \mid X_n = i\} = \frac{p_0^1 i\mu}{i\mu + i(i-1)\,\lambda} + \frac{p_1^2 i(i-1)\,\lambda}{i\mu + i(i-1)\,\lambda},$$

$$\mathbf{P}\{X_{n+1} = i + 1 \mid X_n = i\} = \frac{p_2^1 i\mu}{i\mu + i(i-1)\,\lambda} + \frac{p_3^2 i(i-1)\,\lambda}{i\mu + i(i-1)\,\lambda}.$$

The probability of the event that the chain X_n hits the absorbing state 0 provided that the initial state is i coincides with the extinction probability q_{i0} for the process ξ_t. We have the following relationship for the extinction probabilities:

$$q_{i0} = \frac{p_0^1\mu + p_1^2(i-1)\,\lambda}{\mu + (i-1)\,\lambda}\,q_{i-1,0} + \frac{p_2^1\mu + p_3^2(i-1)\,\lambda}{\mu + (i-1)\,\lambda}\,q_{i+1,0}.$$

Hence, we obtain the linear homogeneous difference equation with linear coefficients (see [9, Chap. 7, section 46, p. 5], and [10, Chap. 2, section 5])

$$(p_3^2\lambda(i+2) - 2p_3^2\lambda + p_2^1\mu)\, q_{i+2,0} - (\lambda(i+1) - \lambda + \mu)\, q_{i+1,0} + (p_1^2\lambda i + p_0^1\mu)\, q_{i0} = 0,$$

$i \in \mathbf{N}$, and the initial condition $q_{00} = 1$.

The solution of the last equation admits the estimate (see [9])

$$|q_{i0}| \leqq C_\varepsilon \left(\frac{1}{R} - \varepsilon\right)^i, \quad i \in \mathbf{N}, \tag{8}$$

where $\varepsilon > 0$, $C_\varepsilon > 0$ is a constant, and $R = \min(|z_1|, |z_2|)$, where z_1 and z_2 are the roots of the equation $p_1^2 z^2 - z + p_3^2 = 0$. Under the conditions of the lemma, $z_1 = 1$, $z_2 = p_3^2/p_1^2 > 1$, and $R = 1$. Now (8) implies the statement of the lemma.

Similarly, applying methods used in [9], one can consider the case $h_2(s) = p_3^2 s^3 + p_1^2 s + p_0^2$. The lemma is proved.

In the case under consideration, equation (5) for the exponential generating function of the extinction probabilities $g_0(z)$ has the form

$$\lambda p_3^2 z g_0'''(z) + (-\lambda z + \mu p_2^1)\, g_0''(z) + (\lambda p_1^2 z - \mu)\, g_0'(z) + (\lambda p_0^2 z + \mu p_0^1)\, g_0(z) = 0. \tag{9}$$

Under our hypotheses we have

$$h_1(s) - s = p_2^1(s-1)(s-q_1), \qquad h_2(s) - s^2 = p_3^2(s-1)(s-q_2)(s-\widehat{q}_2). \tag{10}$$

The conditions $m_1 > 0$ and $p_0^1 > 0$ are equivalent to the condition $0 < q_1 < 1$. In addition, it is not difficult to check that the conditions $m_2 > 0$ and $p_0^2 + p_1^2 > 0$ imply $0 < q_2 < 1$ and $-q_2 < \widehat{q}_2 \leqq 0$ [7].

THEOREM 1. *Let*

$$h_1(s) = p_2^1 s^2 + p_0^1, \qquad h_2(s) = p_3^2 s^3 + p_1^2 s + p_0^2$$

with $m_2 > 0$ *and* $p_0^2 + p_1^2 > 0$. *Assume* $q_1 = q_2$. *Then the extinction probabilities of* ξ_t *are* ($\lambda > 0$, $\mu > 0$): $q_{i0} = q_2^i$, $i \in \mathbf{N}$.

Proof. First we find a solution of equation (9) under condition (6) and $g_0(0) = 1$. The point $z = 0$ is a weak singular point for equation (9) (see [8, Chap. 5, sections 18.2, 18.6(a)]). Following [8] one can show that the linear differential equation (9) of the third order does not have three linearly independent solutions such that each of them is analytical everywhere in the complex z-plane. Direct substitution of the functions $e^{q_2 z}$ and e^z in equation (9) shows that they are particular solutions of this equation. Therefore, according to condition (6) the solution $g_0(z)$ we search for has the form

$$g_0(z) = C_1 e^{q_2 z} + C_2 e^z,$$

where C_1 and C_2 are some constants. Hence we obtain $q_{i0} = C_1 q_2^i + C_2$ and $\lim_{i\to\infty} q_{i0} = C_2$. By the lemma just proved, $C_2 = 0$. Recalling the condition $g_0(0) = 1$, we finally get $g_0(z) = e^{q_2 z}$. Theorem 1 is proved.

THEOREM 2. *Let*

$$h_1(s) = p_2^1 s^2 + p_0^1 \quad \textit{and} \quad h_2(s) = p_3^2 s^3 + p_1^2 s + p_0^2$$

with $p_0^1 + p_0^2 > 0$, $m_2 > 0$, *and* $p_0^2 + p_1^2 > 0$. *Put*

$$\alpha = \frac{\mu}{\lambda} \cdot \frac{h_1(q_2) - q_2}{h_2'(q_2) - 2q_2}, \qquad \beta = \frac{\mu}{\lambda} \cdot \frac{h_1(\widehat{q}_2) - \widehat{q}_2}{h_2'(\widehat{q}_2) - 2\widehat{q}_2} \tag{11}$$

and assume $\alpha > 0$. *The extinction probabilities of the branching process* ξ_t *equal* ($\lambda > 0$, $\mu > 0$),

$$q_{i0} = \frac{1}{A} \int_{\widehat{q}_2}^{q_2} \frac{u^i (q_2 - u)^{\alpha-1} (u - \widehat{q}_2)^{\beta-1}}{p_3^2 (1-u)}\, du, \quad i \in \mathbf{N}, \tag{12}$$

where the constant A *is specified by the condition* $q_{00} = 1$,

$$A = \int_{\widehat{q}_2}^{q_2} \frac{(q_2 - u)^{\alpha-1} (u - \widehat{q}_2)^{\beta-1}}{p_3^2 (1-u)}\, du.$$

Proof. Let us find a solution of equation (9) given (6) and $g_0(0) = 1$. Following [8, Part III, eq. (5.7)], we conclude that ($\widehat{q}_2 < u_0 < q_2$),

$$\begin{aligned} \varphi(u) &= \frac{1}{p_3^2 (u-1)(u-q_2)(u-\widehat{q}_2)} \exp\left(\frac{\mu}{\lambda} \int_{u_0}^{u} \frac{p_2^1 (v-1)(v-q_1)}{p_3^2 (v-1)(v-q_2)(v-\widehat{q}_2)}\, dv \right) \\ &= C_0 \cdot \frac{(q_2-u)^{\alpha-1}(u-\widehat{q}_2)^{\beta-1}}{p_3^2(u-1)}, \qquad C_0 = \frac{1}{(q_2-u_0)^{\alpha}(u_0-\widehat{q}_2)^{\beta}}, \end{aligned}$$

where α and β are specified by (11). Substituting expressions (10) in (11), we obtain

$$\alpha = \frac{\mu}{\lambda} \cdot \frac{p_2^1 (q_2 - q_1)}{p_3^2 (q_2 - \widehat{q}_2)}, \qquad \beta = \frac{\mu}{\lambda} \cdot \frac{p_2^1 (q_1 - \widehat{q}_2)}{p_3^2 (q_2 - \widehat{q}_2)}. \tag{13}$$

Hence it follows that $\beta > 0$.

Given $\alpha > 0$ and $\beta > 0$, one may take as the curve of integration K in (7) the interval between the roots $\widehat{q}_2$ and q_2 of the equation $h_2(s) - s^2 = 0$ (see [8, Part I, Chap. 5, section 19.2]), the last item of point (B) in [8]; as a result we find a particular solution of equation (9),

$$\int_{\widehat{q}_2}^{q_2} e^{zu} \frac{(q_2-u)^{\alpha-1}(u-\widehat{q}_2)^{\beta-1}}{p_3^2(1-u)}\, du$$

(one can check this fact by direct substitution of the last expression in equation (9)).

This particular solution is analytical in the complex z-plane (compare with the proof of the corollary below). Thus, the differential equation (9) of the third order does not have three linearly independent analytical solutions; by virtue of condition (6),

$$g_0(z) = C_1 \int_{\widehat{q}_2}^{q_2} e^{zu} \frac{(q_2-u)^{\alpha-1}(u-\widehat{q}_2)^{\beta-1}}{p_3^2(1-u)}\, du + C_2 e^z,$$

where C_1 and C_2 are some constants. As in the proof of Theorem 1, one can show that $C_2 = 0$. Taking into account the boundary condition $g_0(0) = 1$, we obtain (12). Theorem 2 is proved.

Let us calculate the value of the constant A. Making the change of variable $x = (u - \widehat{q}_2)/(q_2 - \widehat{q}_2)$, we get

$$A = \frac{(q_2-\widehat{q}_2)^{\alpha+\beta-1}}{p_3^2(1-\widehat{q}_2)} \int_0^1 \frac{x^{\beta-1}(1-x)^{\alpha-1}}{1-xc}\, dx = \frac{(q_2-\widehat{q}_2)^{\alpha+\beta-1}}{p_3^2(1-\widehat{q}_2)} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} F(1,\beta;\alpha+\beta;c),$$

where $c = (q_2 - \widehat{q}_2)/(1 - \widehat{q}_2) < 1$; here $\Gamma(\alpha)$ is the gamma-function and $F(\alpha,\beta;\gamma;c)$ is the hypergeometric function.

The same change of variable for the integral (12) gives

$$q_{i0} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)F(1,\beta;\alpha+\beta;c)} \int_0^1 \frac{((q_2-\widehat{q}_2)x+\widehat{q}_2)^i x^{\beta-1}(1-x)^{\alpha-1}}{1-cx}\, dx.$$

Applying to the last integral the standard methods of finding asymptotic formulas as $i \to \infty$ (see, for instance, [11, Thm. 16.1]), we establish the following statement.

COROLLARY. *The following representation is valid as* $i \to \infty$:

$$q_{i0} \sim C \cdot \frac{q_2^i}{i^\alpha}, \qquad C = \frac{1}{1-c} \cdot \frac{q_2^\alpha}{(q_2 - \widehat{q}_2)^\alpha} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)\, F(1,\beta;\alpha+\beta;c)}.$$

Note that $C = 1$ if $\alpha = 0$ (in this case $q_1 = q_2$ by virtue of condition (13)).

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